Lecture 5 Frequency Domain

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Outline

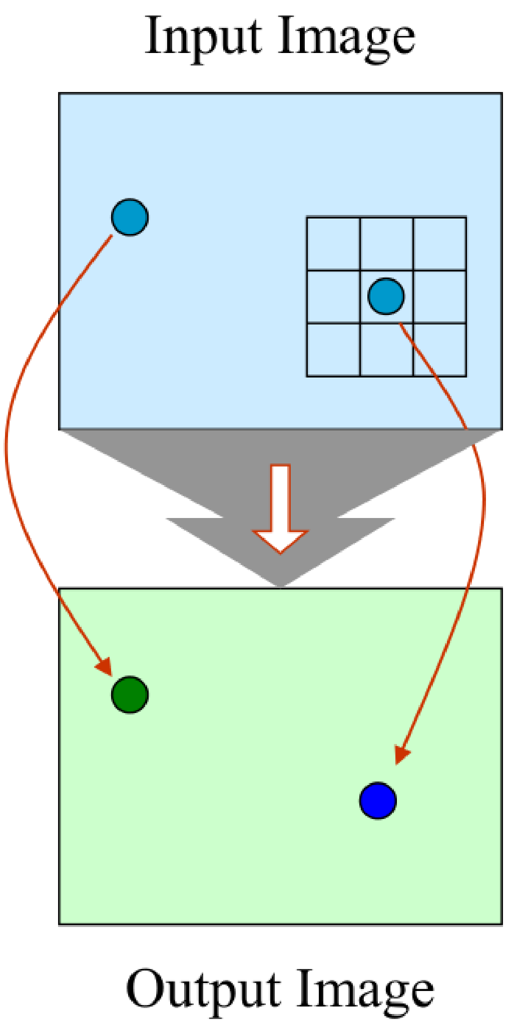
- Review
- Fourier Series
- Fourier Integral Theorem
- Fourier Transform
- Discrete Fourier Transform

Review

❑ Operation Types

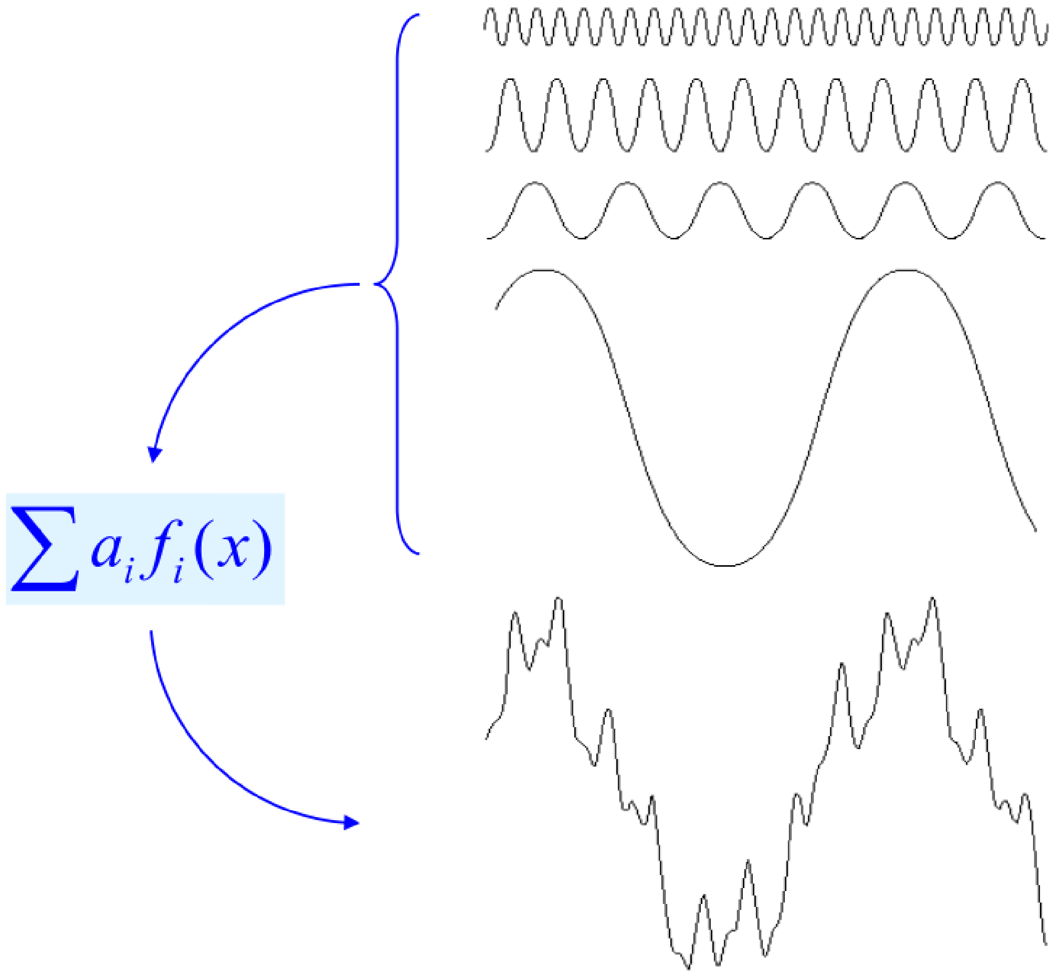
- ❖ Point Operation
  - Gray-level transformation
- ❖ Local Operation
  - Mask Processing or filtering
- ❖ Global Operation
  - Use values of all pixels
  - (e.g.) Fourier transform

    Histogram equalization, etc



Preliminary Concepts

- ❑ Is it possible?



Fourier Series

- Fourier Series corresponding to a function f(x)
  - Which is defined in the interval $c \le x \le c + 2L$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

Fourier Series

where

$$a_n = \frac{1}{L}\int_{c}^{c+2L} f(x)\cos\frac{n\pi x}{L}dx$$

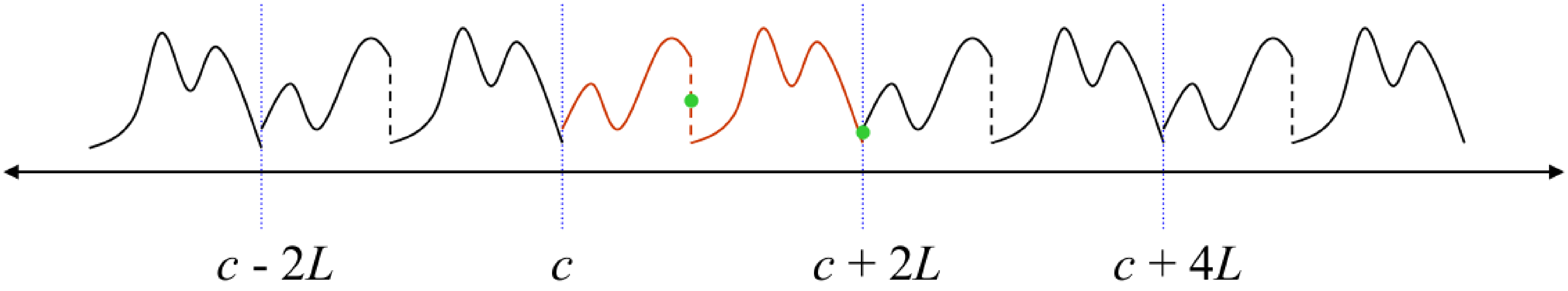
$$b_n = \frac{1}{L}\int_{c}^{c+2L} f(x)\sin\frac{n\pi x}{L}dx$$

❑ Fourier Series *vs*. Periodic Extension of f(x)

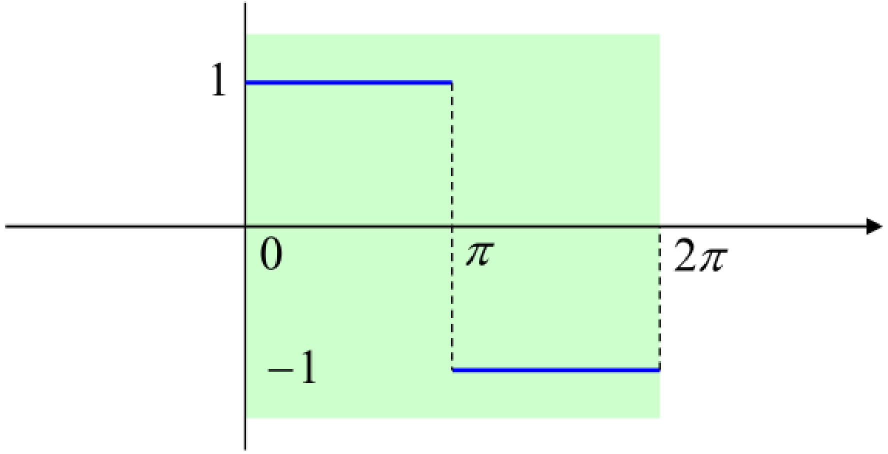
- If f(x) and df(x)/dx are piecewise continuous and f(x) is defined by periodic extension of period 2L, i.e.

$$f(x+2L)=f(x)$$

then the series converges to f(x) if x is a point of continuity and to (1/2){f(x+0)+f(x-0)} if x is a point of discontinuity.



❑ Example 1



$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos\frac{0\pi x}{\pi}dx = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos\frac{n\pi x}{\pi}dx = \frac{1}{\pi}\int_0^{\pi}\cos nx dx - \frac{1}{\pi}\int_{\pi}^{2\pi}\cos nx dx$$

$$= \frac{1}{n\pi}\left[\sin nx\right]_0^{\pi} - \frac{1}{n\pi}\left[\sin nx\right]_{\pi}^{2\pi} = 0$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}dx$$

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)$$

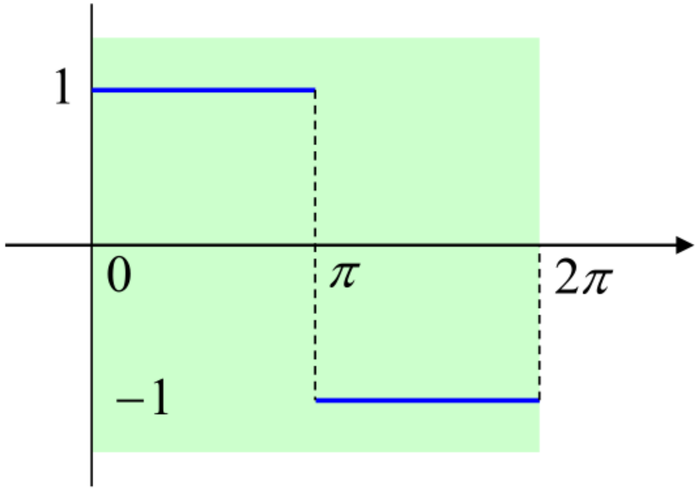
where

$$a_n=\frac{1}{L}\int_c^{c+2L}f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n=\frac{1}{L}\int_c^{c+2L}f(x)\sin\frac{n\pi x}{L}dx$$

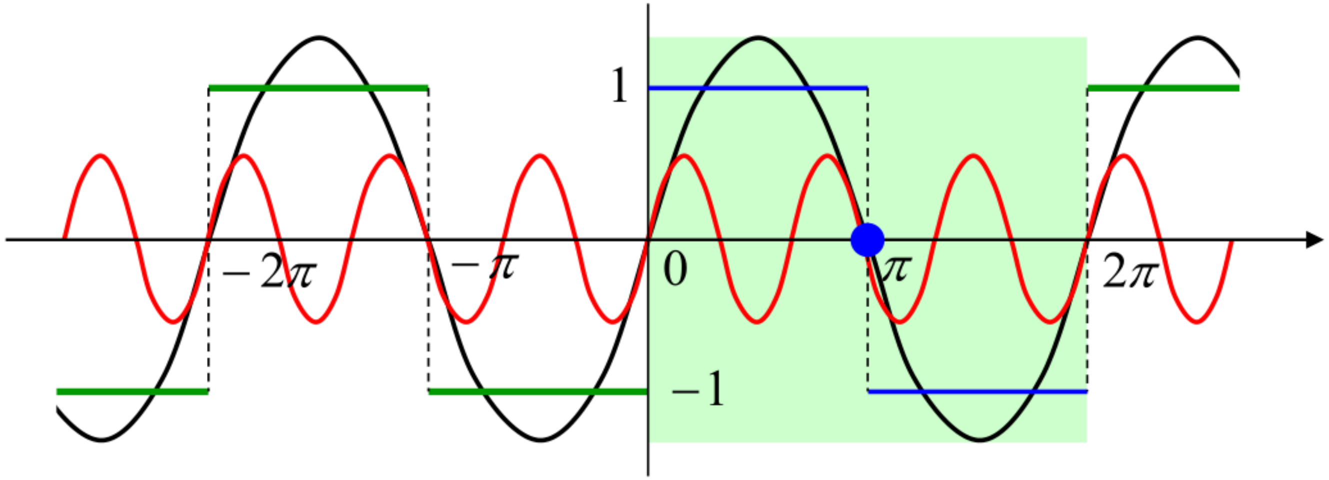
$$b_{2k}=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin\frac{2k\pi x}{\pi}dx=\frac{1}{\pi}\int_0^{\pi}\sin 2kx dx-\frac{1}{\pi}\int_{\pi}^{2\pi}\sin 2kx dx$$

$$=\frac{-1}{2k\pi}\left[\cos 2kx\right]_0^{\pi}+\frac{1}{2k\pi}\left[\cos 2kx\right]_{\pi}^{2\pi}=0$$



$$b_{2k-1}=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin\frac{(2k-1)\pi x}{\pi}dx$$

$$=\frac{-1}{(2k-1)\pi}\left[\cos(2k-1)x\right]_0^{\pi}+\frac{1}{(2k-1)\pi}\left[\cos(2k-1)x\right]_{\pi}^{2\pi}=\frac{4}{(2k-1)\pi}$$



$$\frac{4}{\pi}\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\cdots\right)$$

$$\cong 1.27\sin x+0.42\sin 3x+0.25\sin 5x+\cdots$$

❑ Complex Form

$$f(x)=\sum_{n=-\infty}^{\infty} c_n \exp\left[\frac{jn\pi x}{L}\right]$$

$$c_n=\frac{1}{2L}\int_c^{c+2L} f(x)\exp\left[-\frac{jn\pi x}{L}\right]dx=\begin{cases}(1/2)(a_n-jb_n) & n>0\\ (1/2)(a_{-n}+jb_{-n}) & n<0\\ (1/2)a_0 & n=0\end{cases}$$

Fourier Integral Theorem

- Continuous Form of Fourier Series

$$f(x) = \int_0^{\infty} \{A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\} d\alpha$$

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\alpha x\, dx, \quad B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\alpha x\, dx$$

Sufficient conditions under which this theorem holds are:

(1) $f(x)$ and $df(x)/dx$ are piecewise continuous in every finite interval $-L < x < L$;

(2) $\int_{-\infty}^{\infty} |f(x)| dx$ converges;

(3) $f(x)$ is replaced by $(1/2)\{f(x+0)+f(x-0)\}$ if x is a point of discontinuity.

Fourier Integral Theorem

$$f(x) = \int_0^\infty \{A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\} d\alpha$$

$$\frac{1}{\pi}\left\{\left(\int_{-\infty}^{\infty} f(u)\cos\alpha u\, du\right)\cos\alpha x + \left(\int_{-\infty}^{\infty} f(u)\sin\alpha u\, du\right)\sin\alpha x\right\}$$

$$= \frac{1}{\pi}\left(\int_{-\infty}^{\infty} f(u)\cos\alpha u\cos\alpha x\, du + \int_{-\infty}^{\infty} f(u)\sin\alpha u\sin\alpha x\, du\right)$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)(\cos\alpha u\cos\alpha x + \sin\alpha u\sin\alpha x)du$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\alpha(x-u)\, du$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\cos\alpha(x-u)\, du\, d\alpha$$

Fourier Integral Theorem

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\cos\alpha(x-u)\,du\,d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left(\int_{-\infty}^{\infty}\cos\alpha(x-u)\ d\alpha\right)du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty}\left(\cos\alpha(x-u) + j\sin\alpha(x-u)\right)d\alpha\right]du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty}\exp[j\alpha(x-u)]\,d\alpha\right]du \qquad j=\sqrt{-1}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\exp[j\alpha(x-u)]\,du\,d\alpha$$

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$ Euler's Identity

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\exp[j\alpha(x-u)]du\,d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(u)\exp[-j\alpha u]du\right)\exp[j\alpha x]d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\exp[j\alpha x]d\alpha$$

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\exp[-j\omega x]dx$$

Fourier Transform

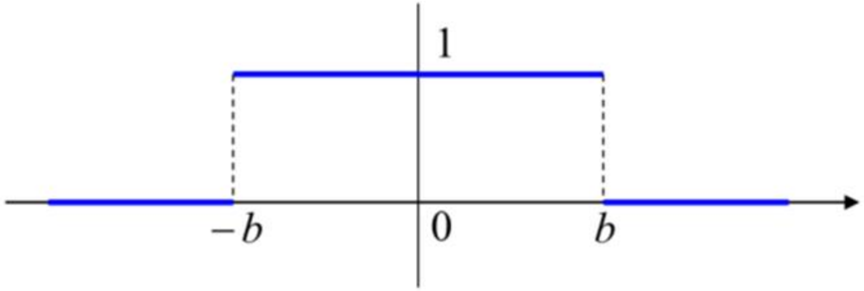
- Fourier Transform

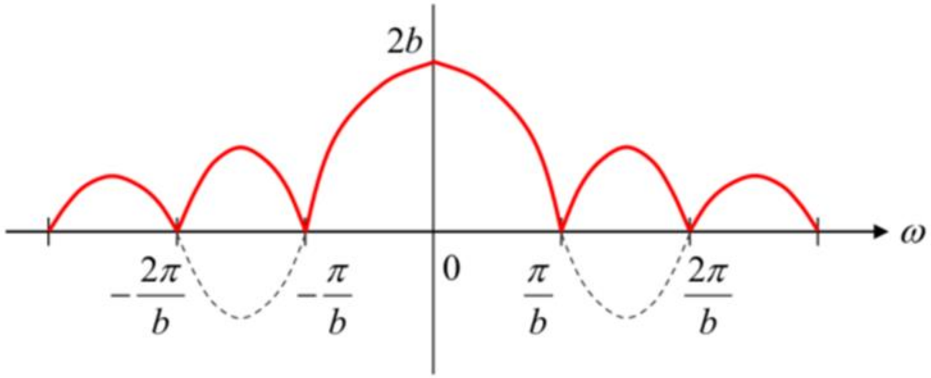
$$\Im\{f(x)\} = F(\omega) = \int_{-\infty}^{\infty} f(x)\exp[-j\omega x]dx$$

$$\Im^{-1}\{F(\omega)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp[j\omega x]d\omega$$

❑ Example 1

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\exp[-j\omega x]dx = \int_{-b}^{b} \exp[-j\omega x]dx = \int_{-b}^{b} (\cos\omega x - j\sin\omega x)dx = \frac{1}{\omega}[\sin\omega x]_{-b}^{b} = \frac{2}{\omega}\sin b\omega$$



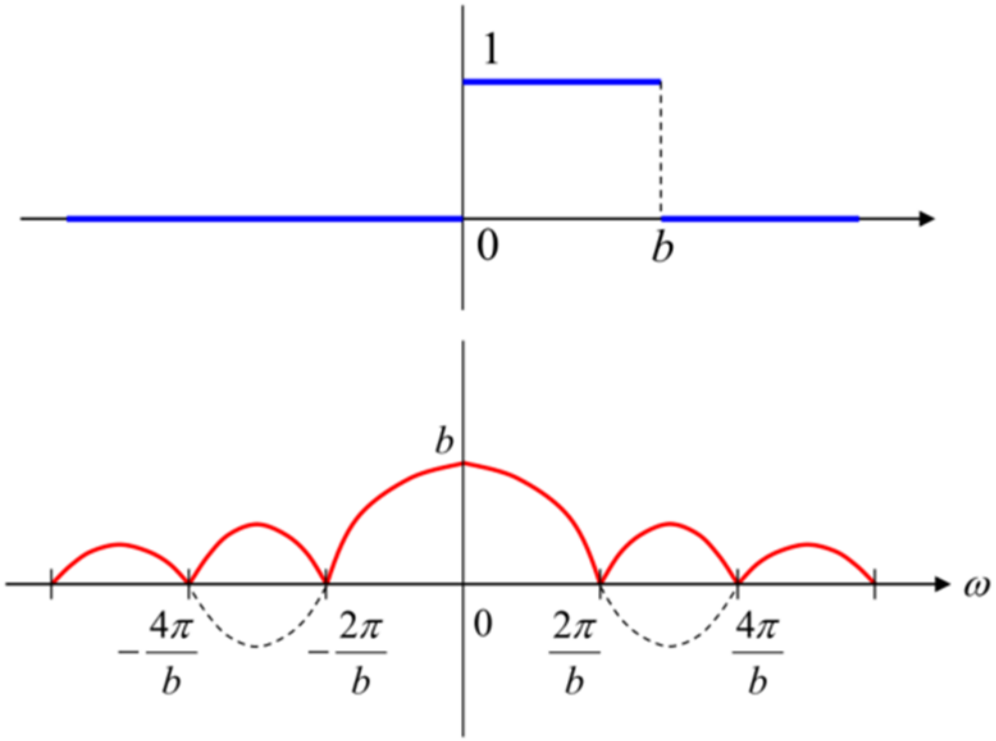


$$|F(\omega)| = 2b\left|\frac{\sin b\omega}{b\omega}\right|$$

$\frac{\sin b\omega}{b\omega}$ Sinc Function

Magnitude of complex number

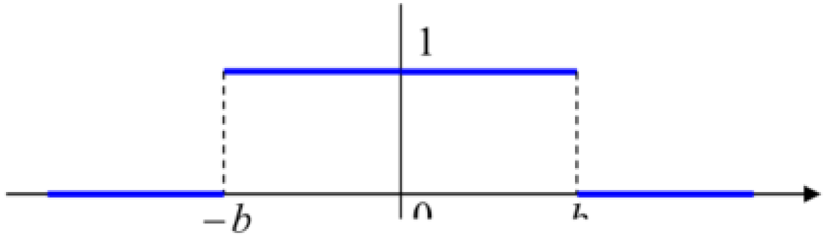
❑ Example 2

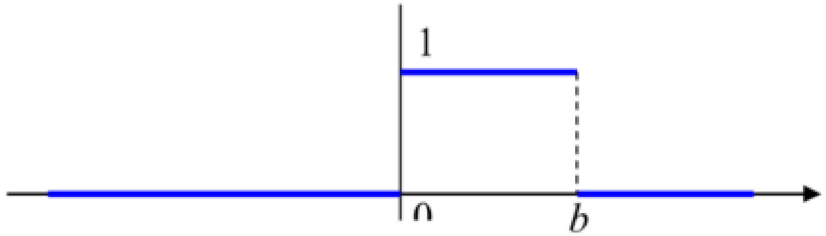


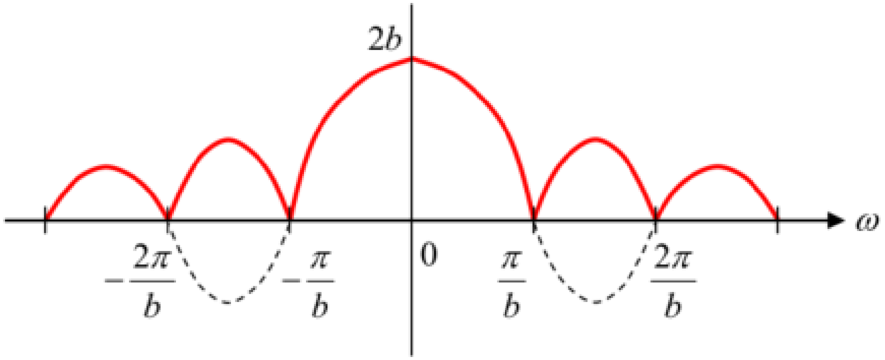
$$
\begin{aligned}
F(\omega)
&= \int_0^b \exp[-j\omega x]dx \\
&= \frac{-1}{j\omega}\left[\exp(-j\omega x)\right]_0^b \\
&= \frac{1}{j\omega}\left[1-\exp(-jb\omega)\right] \\
&= \frac{\exp(-jb\omega/2)}{j\omega}\left[\exp\left(\frac{jb\omega}{2}\right)-\exp\left(-\frac{jb\omega}{2}\right)\right] \\
&= b\frac{\sin(b/2)\omega}{(b/2)\omega}\exp(-jb\omega/2)
\end{aligned}
$$

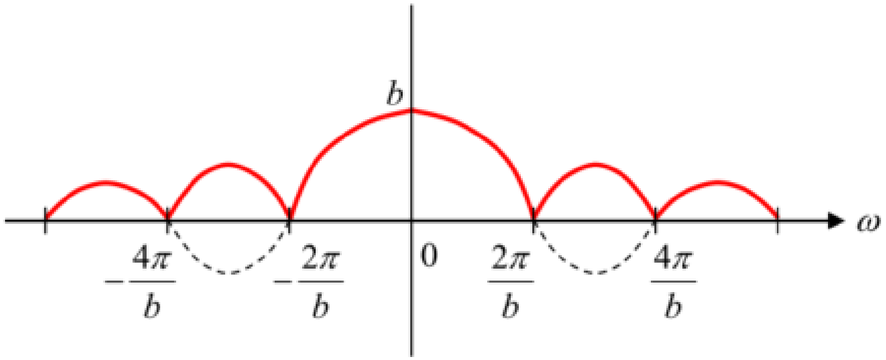
$$
|F(\omega)| = b\left|\frac{\sin(b/2)\omega}{(b/2)\omega}\right|
$$

Fourier Transform









$$|F(\omega)| = 2b\left|\frac{\sin b\omega}{b\omega}\right|$$

$$|F(\omega)| = b\left|\frac{\sin(b/2)\omega}{(b/2)\omega}\right|$$

❑ Another form of Fourier Transform

$$F(u) = \int_{-\infty}^{\infty} f(x)\exp[-j2\pi ux]dx$$

$$f(x) = \int_{-\infty}^{\infty} F(u)\exp[j2\pi ux]du$$

$$\omega = 2\pi u \quad \Rightarrow \quad d\omega = 2\pi\, du$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp[j\omega x]d\omega$$

$$= \int_{-\infty}^{\infty} F(u)\exp[j2\pi ux]du$$

❑ Frequency Domain *vs*. Euler's Formula

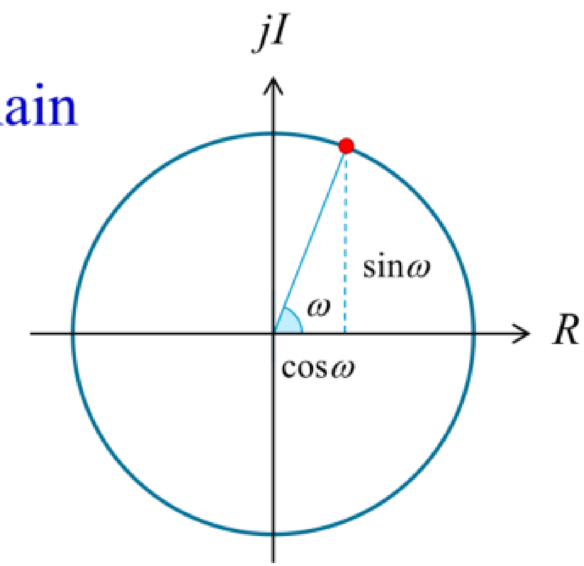
$$e^{j\omega} = \cos\omega + j\sin\omega \quad \Rightarrow \quad T = 2\pi \quad for \;\; \omega$$

$$e^{j2\pi u} = \cos 2\pi u + j\sin 2\pi u \quad \Rightarrow \quad T = 1 \quad for \;\; u$$

- The variable u represents frequency of the transform kernel
- So u is called a frequency variable and the u domain is called a frequency domain

$$a + jb = m\,e^{j\theta} \quad magnitude \;\; m = \sqrt{a^2 + b^2}$$

$$phase \;\; \theta = \tan^{-1}(b/a)$$



Discrete Fourier Transform

❑ Discrete Form of Fourier Transform

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\exp[-j2\pi ux/M]$$

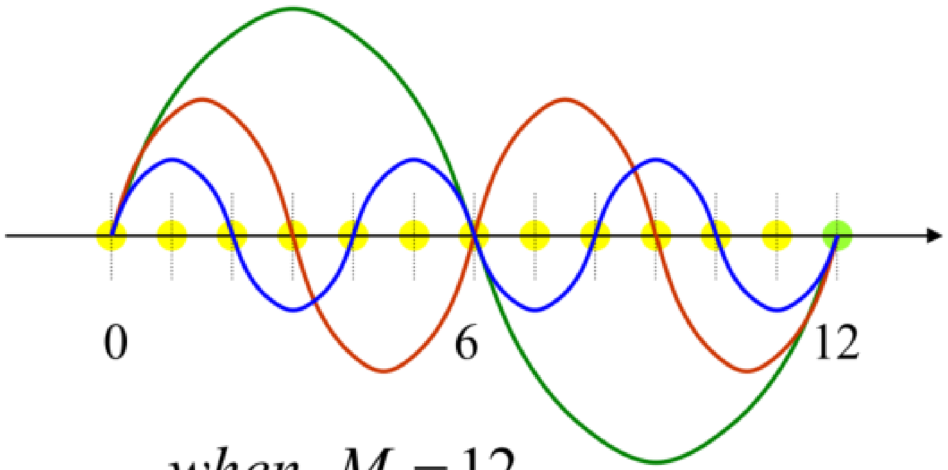
$$f(x) = \sum_{u=0}^{M-1} F(u)\exp[j2\pi ux/M]$$

M : number of samples

$u, x = 0,1,2,\ldots,M-1$

$$\sin\frac{2\pi u}{M}x$$

$$\Rightarrow \quad T = \frac{M}{u} \quad for \quad x$$

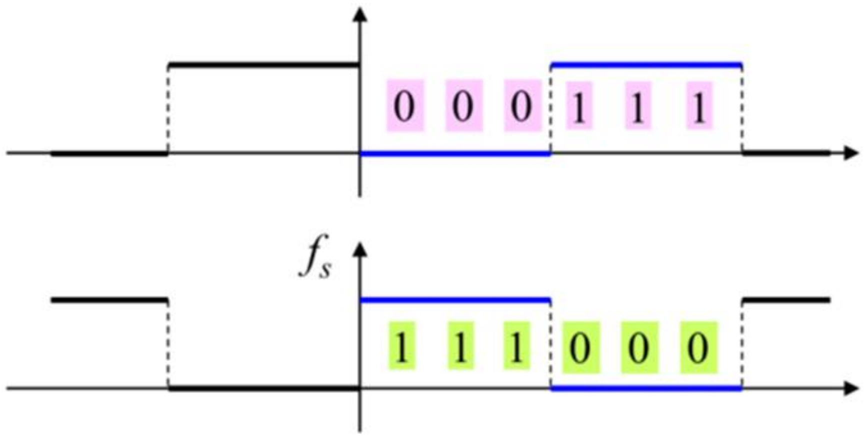


$when \quad M = 12$

- Example

$$F(u)=\frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp[-j2\pi ux/N]$$

$$f(x)=\sum_{u=0}^{N-1} F(u)\exp[j2\pi ux/N]$$



?

f	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1	
$\lvert F \rvert$	3	2	0	1	0	2	3	2	0	1	0	2	3	2	0	1	0	2	← (1/6)

$$F[\ 0\ 1\ 2\ 3\ 4\ 5\ ]=(1/6)\left[\ 3\quad -1+j\sqrt{3}\quad 0\quad -1\quad 0\quad -1-j\sqrt{3}\ \right]$$

$$|F|[\ 0\ 1\ 2\ 3\ 4\ 5\ ]=(1/6)\left[\ 3\ 2\ 0\ 1\ 0\ 2\ \right]$$

$$F_s[\ 0\ 1\ 2\ 3\ 4\ 5\ ]=(1/6)\left[\ 3\quad 1-j\sqrt{3}\quad 0\quad 1\quad 0\quad 1+j\sqrt{3}\ \right]$$

Discrete Fourier Transform

$$f_1[0\ 1\ 2\ 3\ 4\ 5] = [0\quad 0\quad 0\quad 1\quad 1\quad 1]$$

$$f_2[0\ 1\ 2\ 3\ 4\ 5] = [1\quad 1\quad 1\quad 0\quad 0\quad 0]$$

$$f_3[0\ 1\ 2\ 3\ 4\ 5] = [1\quad 1\quad 0\quad 0\quad 0\quad 1]$$

$$f_4[0\ 1\ 2\ 3\ 4\ 5] = [1\quad 0\quad 1\quad 0\quad 1\quad 0]$$

$$f_5[0\ 1\ 2\ 3\ 4\ 5] = [1\quad 1\quad 0\quad 1\quad 1\quad 0]$$

$$|F_{1,2,3}| = (1/6)[3\quad 2\quad 0\quad 1\quad 0\quad 2]$$

$$|F_4| = (1/6)[3\quad 0\quad 0\quad 3\quad 0\quad 0]$$

$$|F_5| = (1/6)[4\quad 0\quad 2\quad 0\quad 2\quad 0]$$

$$F_1[0\ 1\ 2\ 3\ 4\ 5] = (1/6)[3\quad -1+j\sqrt{3}\quad 0\quad -1\quad 0\quad -1-j\sqrt{3}]$$

$$F_2[0\ 1\ 2\ 3\ 4\ 5] = (1/6)[3\quad 1-j\sqrt{3}\quad 0\quad 1\quad 0\quad 1+j\sqrt{3}]$$

$$F_3[0\ 1\ 2\ 3\ 4\ 5] = (1/6)[3\quad 2\quad 0\quad -1\quad 0\quad 2]$$

$$F_4[0\ 1\ 2\ 3\ 4\ 5] = (1/6)[3\quad 0\quad 0\quad 3\quad 0\quad 0]$$

$$F_5[0\ 1\ 2\ 3\ 4\ 5] = (1/6)[4\quad 0\quad 1-j\sqrt{3}\quad 0\quad 1+j\sqrt{3}\quad 0]$$

- DFT and its inverse always exist

$$\sum_{u=0}^{M-1} F(u)\exp[j2\pi ux/M] \qquad F(u)=\frac{1}{M}\sum_{x=0}^{M-1} f(x)\exp[-j2\pi ux/M]$$

$$=\sum_{u=0}^{M-1}\left(\frac{1}{M}\sum_{s=0}^{M-1} f(s)\exp[-j2\pi us/M]\right)\exp[j2\pi ux/M]$$

$$=\frac{1}{M}\sum_{s=0}^{M-1}\sum_{u=0}^{M-1} f(s)\exp[-j2\pi us/M]\exp[j2\pi ux/M]$$

$$=\frac{1}{M}\sum_{s=0}^{M-1} f(s)\left(\sum_{u=0}^{M-1}\exp[j2\pi(x-s)u/M]\right)=\frac{1}{M}(f(x)M)=f(x)$$

$$\because \sum_{u=0}^{M-1}\exp[j2\pi(x-s)u/M]=\begin{cases} M & if\ x=s \\ 0 & otherwixe \end{cases}$$

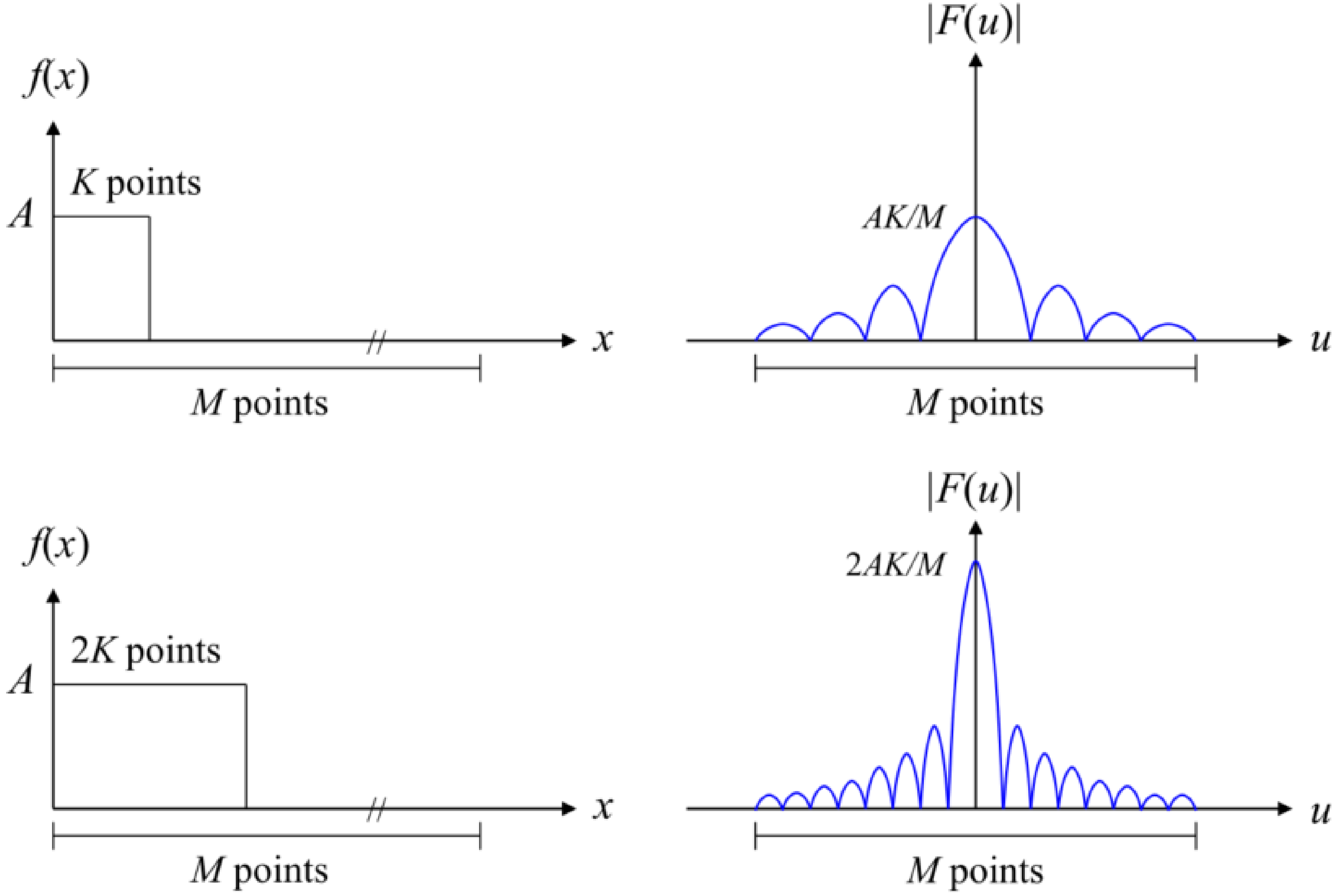
Discrete Fourier Transform

$$F(u) = R(u) + jI(u) = |F(u)|e^{j\phi(u)}$$

- Spectrum $\quad |F(u)| = \left[R^2(u) + I^2(u)\right]^{1/2}$
- Phase Angle $\quad \phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$
- Power Spectrum (Spectral Density)

$$P(u) = |F(u)|^2 = R^2(u) + I^2(u)$$

- Example



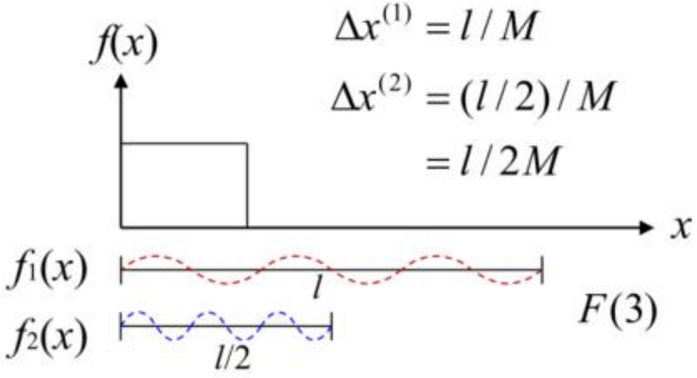
f(x)
K points
A
x
M points
|F(u)|
AK/M
u
M points
f(x)
2K points
A
x
M points
|F(u)|
2AK/M
u
M points

- Relationship between samples in spatial and freq. domain

$f(x_0)$	$f(x_0+\Delta x)$	$f(x_0+2\Delta x)$	$f(x_0+(M-1)\Delta x)$
$f(0)$	$f(1)$	$f(2)$	$f(M-1)$
$F(0)$	$F(1)$	$F(2)$	$F(M-1)$
$F(0)$	$F(\Delta u)$	$F(2\Delta u)$	$F((M-1)\Delta u)$

$$\Delta x^{(1)} = l/M$$

$$\Delta x^{(2)} = (l/2)/M = l/2M$$



$F(3)$

M sample points

$$frequency = f = \frac{1}{T} = \frac{1}{Period}$$

$$u_k^{(1)} = \frac{1}{T_k^{(1)}} = \frac{1}{l/k} = \frac{k}{l} \rightarrow \Delta u^{(1)} = \frac{1}{l}$$

$$u_k^{(2)} = \frac{1}{T_k^{(2)}} = \frac{1}{(l/2)/k} = \frac{2k}{l} \rightarrow \Delta u^{(2)} = \frac{2}{l}$$

$$\therefore \ \Delta u = \frac{1}{M\Delta x}$$

Summary

- ❑ Review
- ❑ Fourier Series
- ❑ Fourier Integral Theorem
- ❑ Fourier Transform
- ❑ Discrete Fourier Transform

❑Next: Filtering in the Frequency Domain

Q & A

Thank You!