Lecture 6 Filtering in Frequency Domain

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Outline

- Fourier Transform
- Filtering in Frequency Domain
- Smoothing Frequency Domain Filters
- Sharpening Frequency Domain Filters

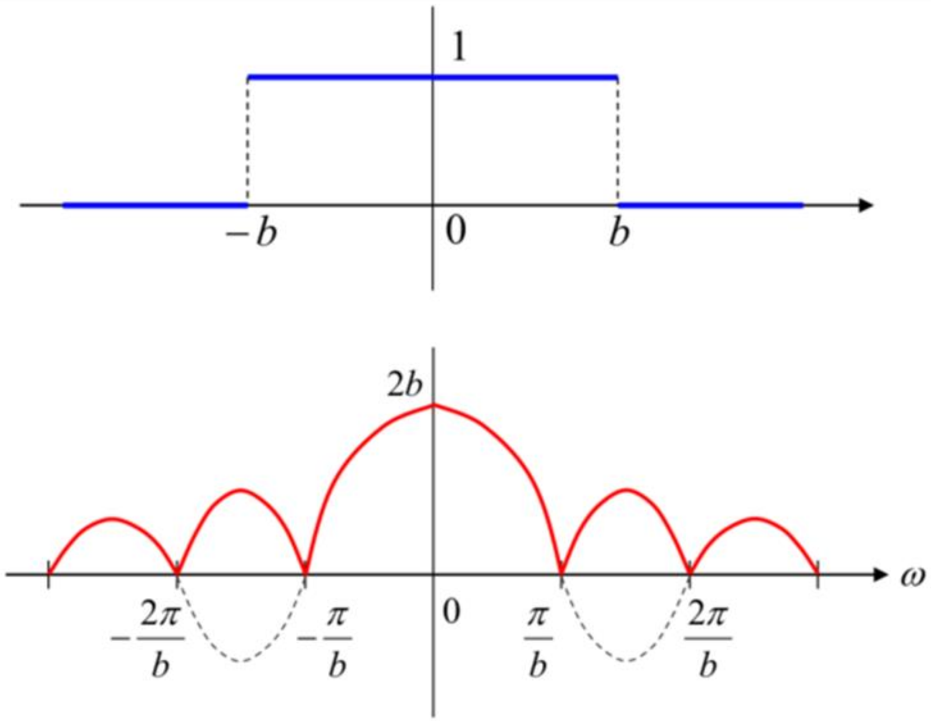
Fourier Transform

❑ 1D Fourier Transform

$$\Im\{f(x)\} = F(\omega) = \int_{-\infty}^{\infty} f(x)\exp[-j\omega x]dx$$

$$\Im^{-1}\{F(\omega)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp[j\omega x]d\omega$$

❑ Example 1



$$
\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(x)\exp[-j\omega x]dx \\
&= \int_{-b}^{b} \exp[-j\omega x]dx \\
&= \int_{-b}^{b} (\cos\omega x - j\sin\omega x)dx \\
&= \frac{1}{\omega}[\sin\omega x]_{-b}^{b} = \frac{2}{\omega}\sin b\omega
\end{aligned}
$$

$$|F(\omega)| = 2b\left|\frac{\sin b\omega}{b\omega}\right|$$

$\frac{\sin b\omega}{b\omega}$ Sinc Function

Magnitude of complex number

-

$$F(u)=\int_{-\infty}^{\infty} f(x)\exp[-j2\pi ux]dx$$

$$f(x)=\int_{-\infty}^{\infty} F(u)\exp[j2\pi ux]du$$

$$\omega = 2\pi u \quad \Rightarrow \quad d\omega = 2\pi\, du$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp[j\omega x]d\omega$$

$$=\int_{-\infty}^{\infty} F(u)\exp[j2\pi ux]du$$

❑ 2D Fourier Transform

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)}\, dx\, dy$$

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)}\, du\, dv$$

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M+vy/N)}$$

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v) e^{j2\pi(ux/M+vy/N)}$$

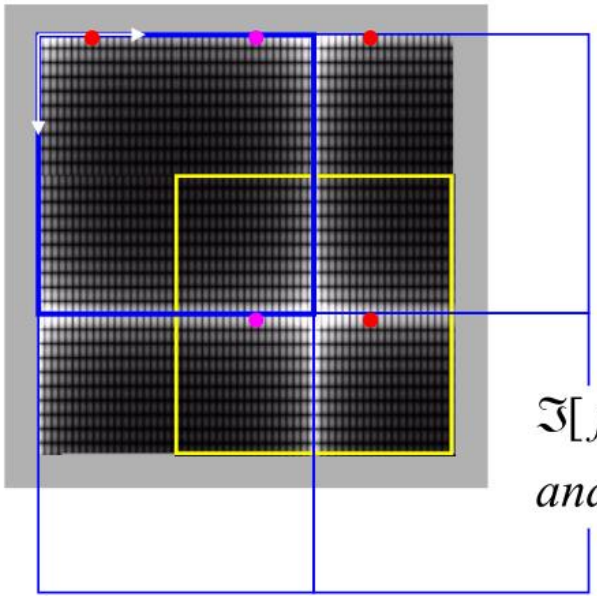
❑ Properties of 2D Fourier Transform

$$\text{If } f(x,y) \text{ is real, } F(u,v) = F^*(-u,-v)$$

$$|F(u,v)| = |F(-u,-v)|$$

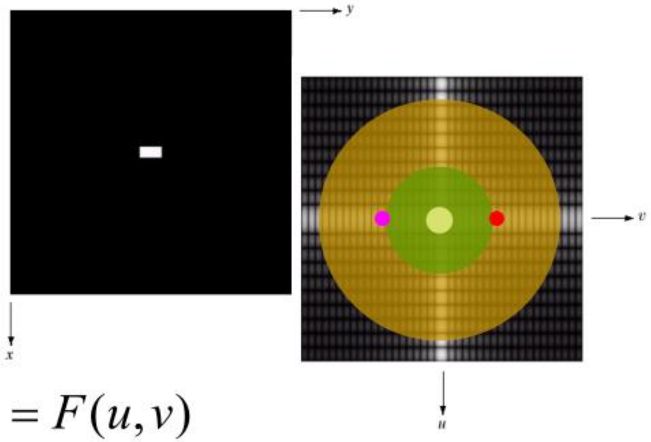
$$\Im\left[f(x,y)(-1)^{x+y}\right] = F(u-M/2, v-N/2)$$

$$\Delta u = \frac{1}{M\Delta x} \quad \& \quad \Delta v = \frac{1}{N\Delta y}$$

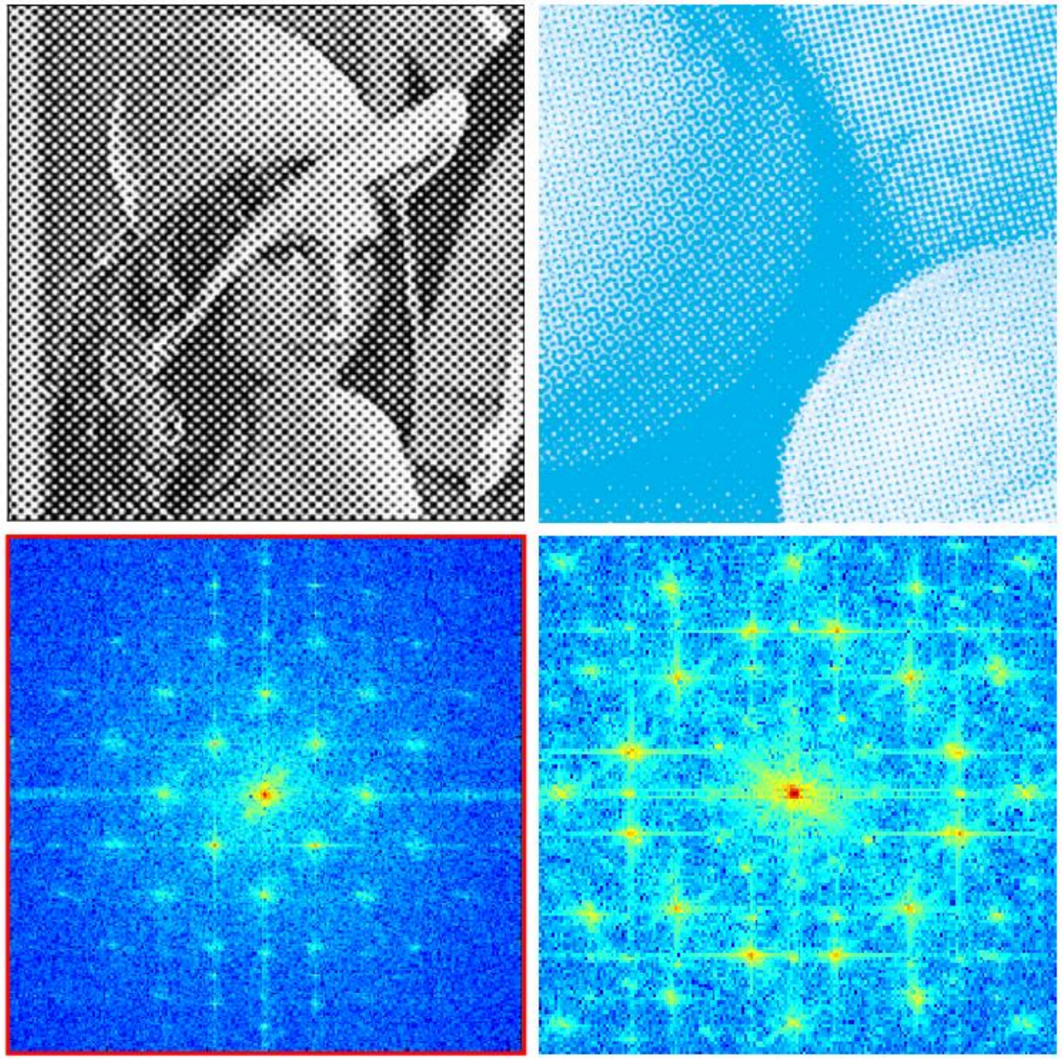


$$\Im[f(x,y)] = F(u,v)$$

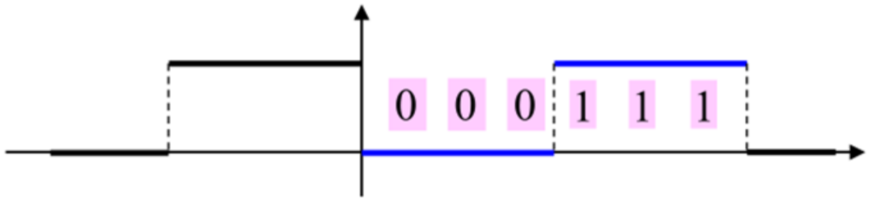
$$and \quad s = c\log(1+r)$$



Fourier Transform



- 1D Case

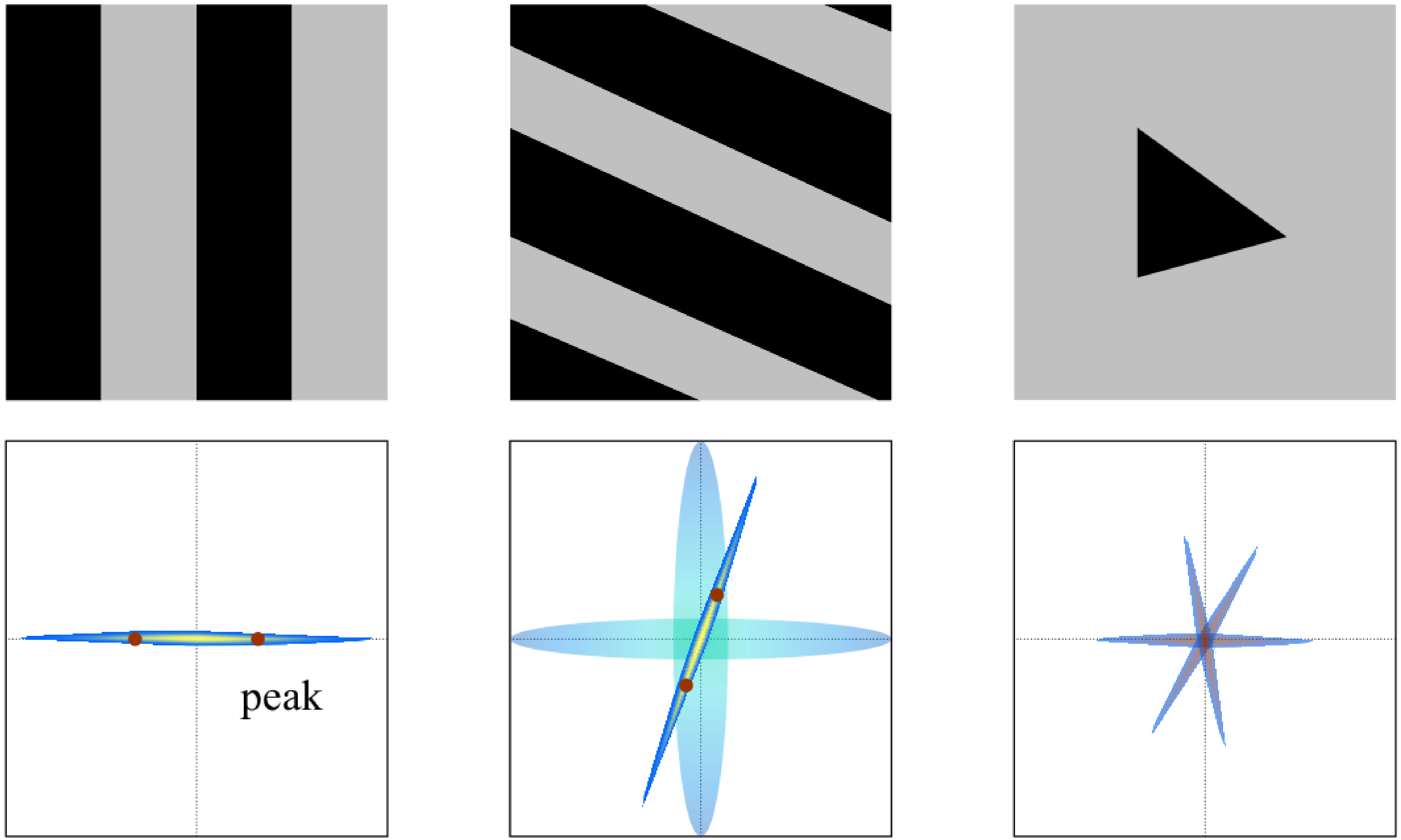


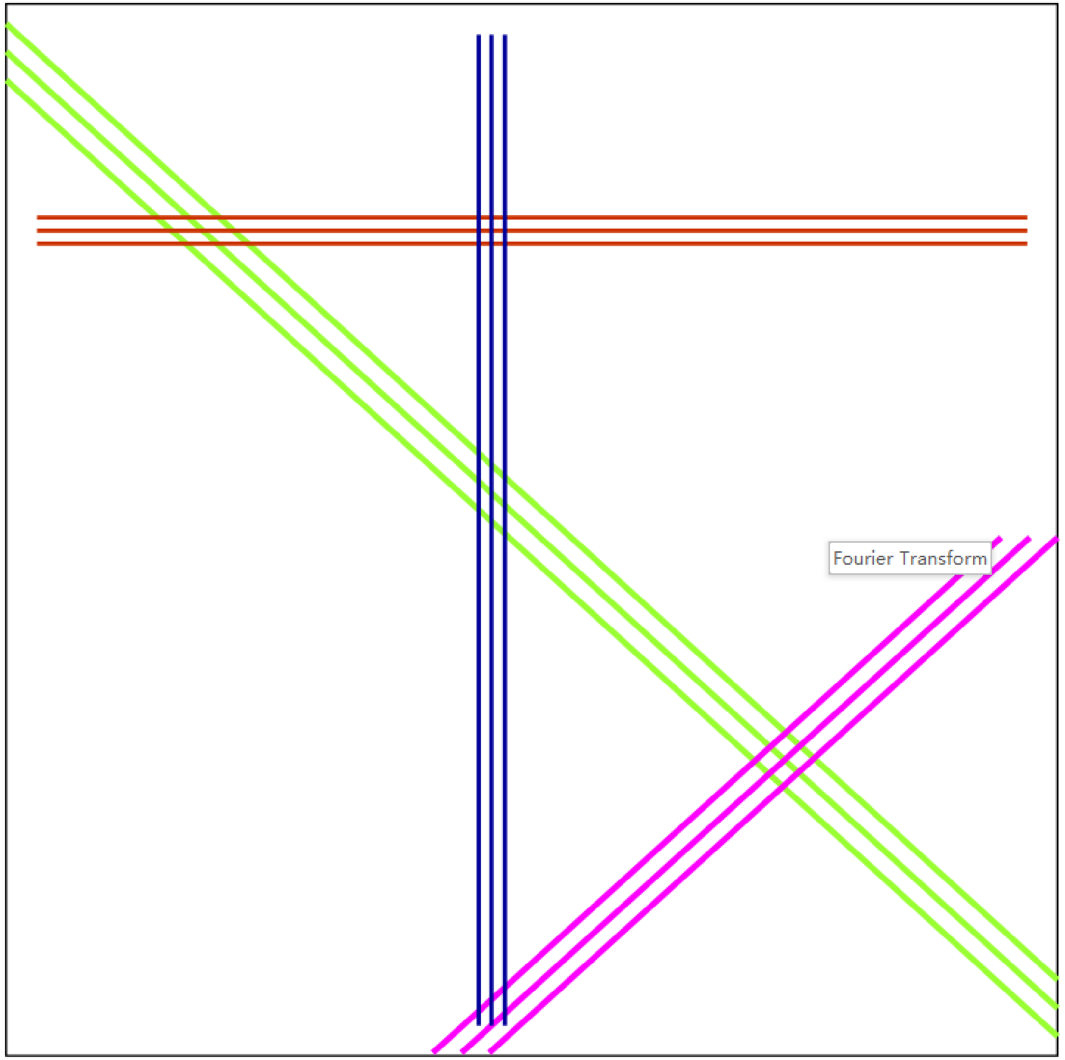
f	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1	
$\lvert F \rvert$	3	2	0	1	0	2	3	2	0	1	0	2	3	2	0	1	0	2	← (1/6)

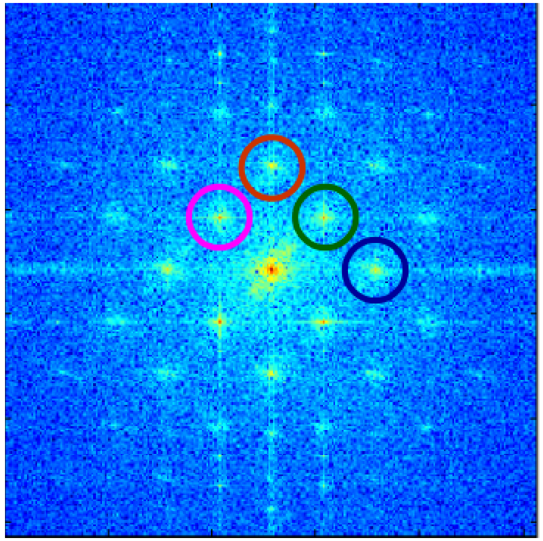
$$F[\;0\;1\;2\;3\;4\;5\;] = (1/6)\left[\;3 \quad -1+j\sqrt{3} \quad 0 \quad -1 \quad 0 \quad -1-j\sqrt{3}\;\right]$$

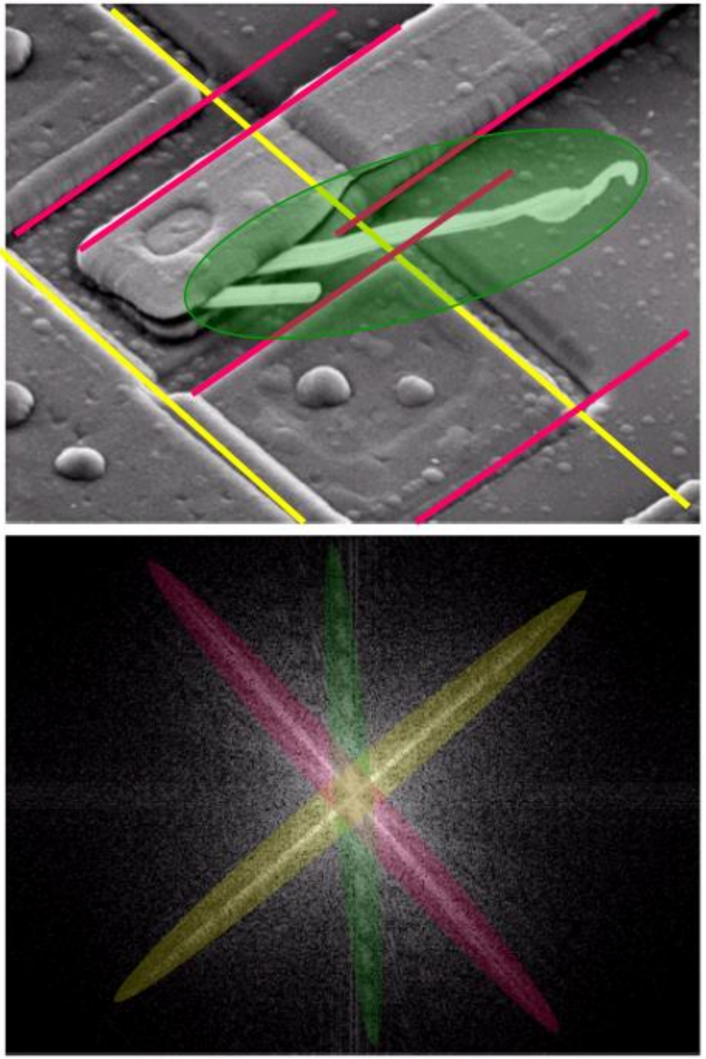
$$|F|[\;0\;1\;2\;3\;4\;5\;] = (1/6)\left[\;3\;2\;0\;1\;0\;2\;\right]$$

$$\left[\;3 \quad -1+j\sqrt{3} \quad 0 \quad -1 \quad 0 \quad -1-j\sqrt{3}\;\right]\left[\;3 \quad -1+j\sqrt{3} \quad 0 \quad -1 \quad 0 \quad -1-j\sqrt{3}\;\right]$$


peak

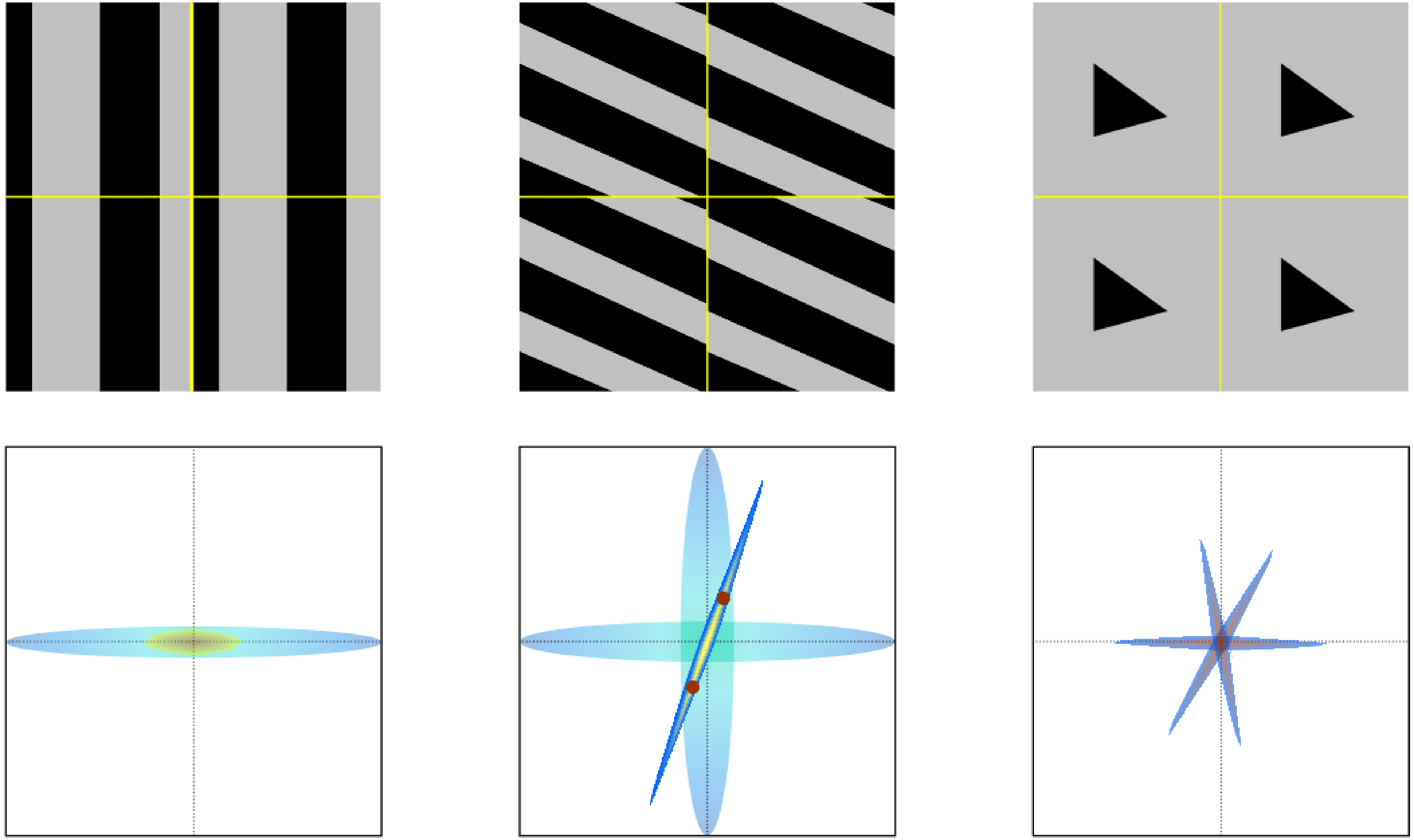

Fourier Transform



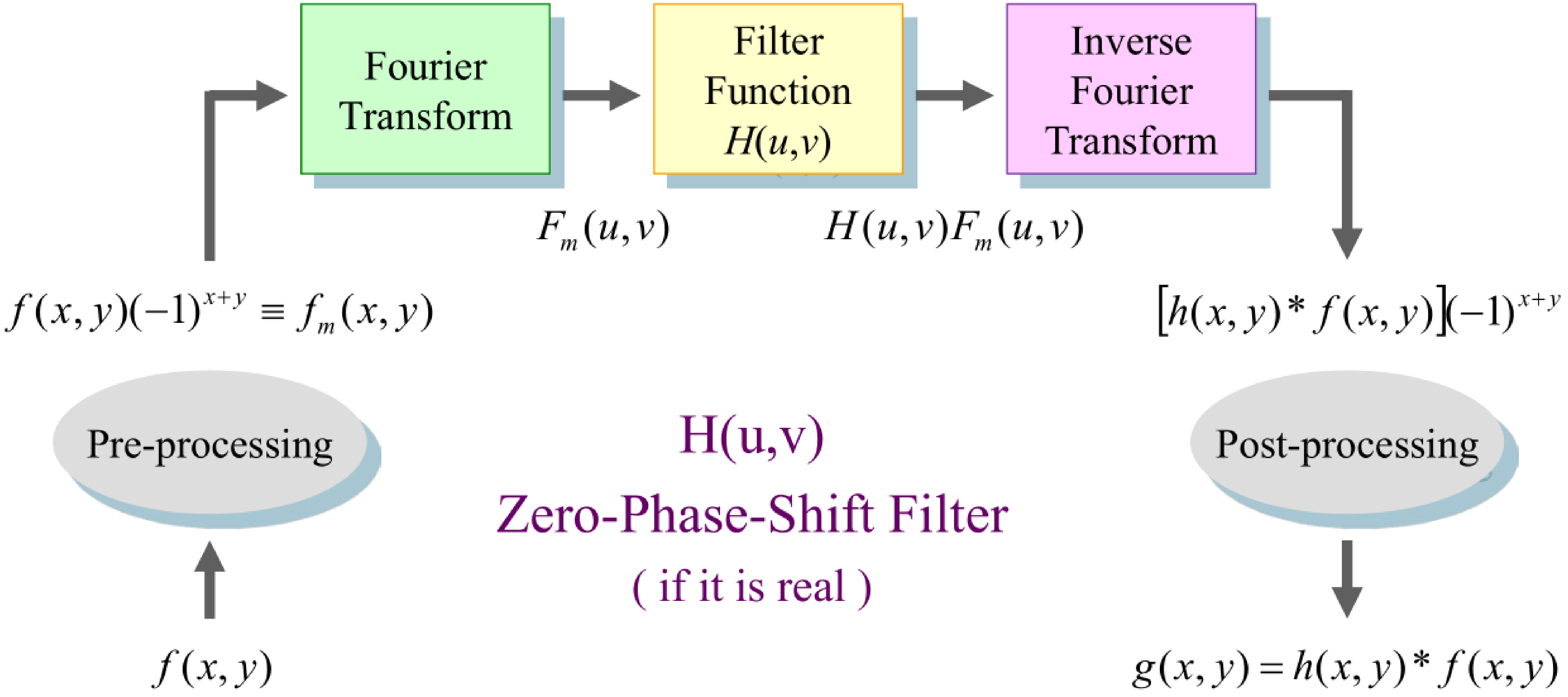


Note the zeros in vertical frequency components, corresponding to narrow span of the white protrusion

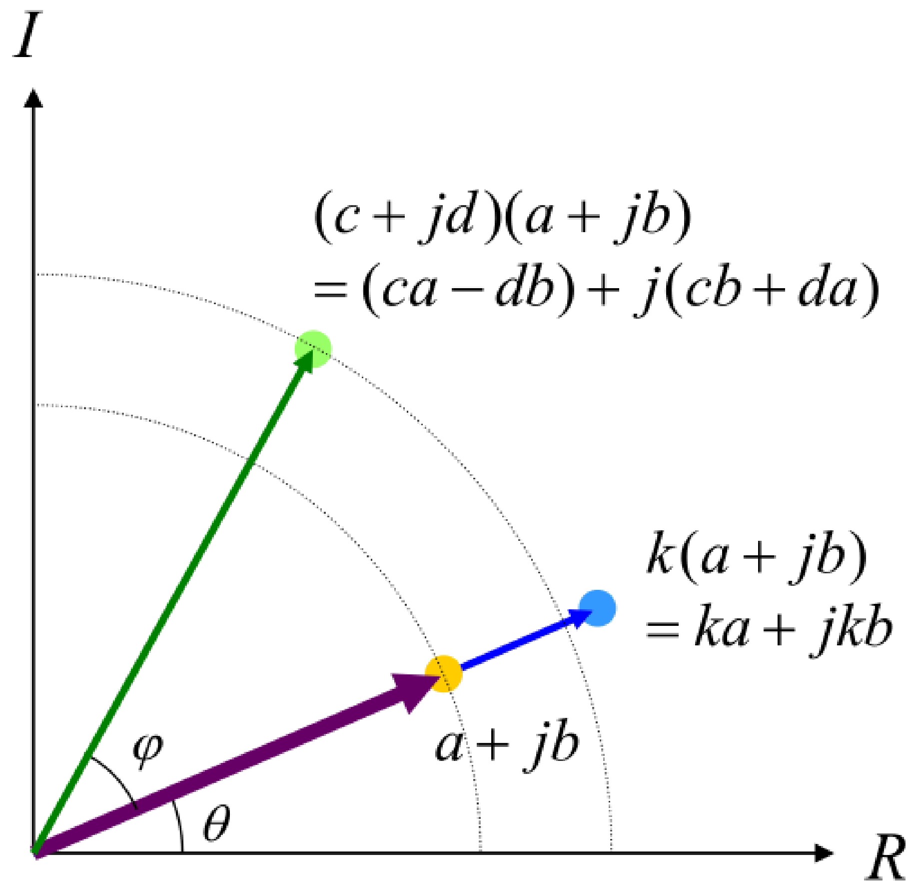
Fourier Transform



Filtering in Frequency Domain



Zero-Phase-Shift ?



$$a + jb = m_1 e^{j\theta}$$

$$c + jd = m_2 e^{j\varphi}$$

$$k(a+jb) = k m_1 e^{j\theta} \quad k > 0$$

$$(c+jd)(a+jb) = m_1 m_2 e^{j(\theta+\varphi)}$$

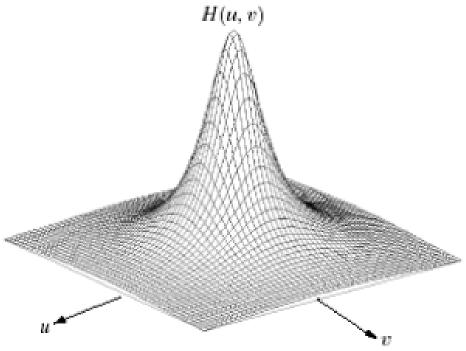
Low Frequency

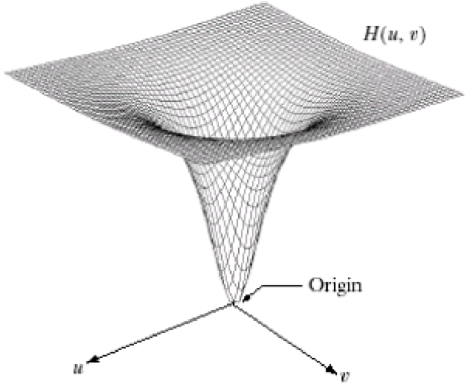
Responsible for general gray-level appearance of an image over smooth areas

High Frequency

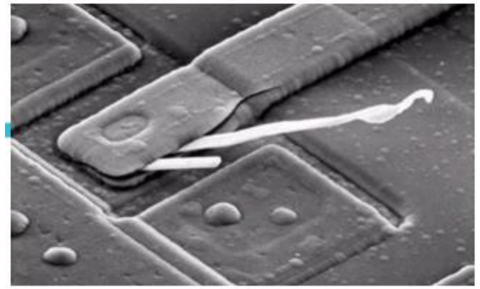
Responsible for detail, such as edges and noise

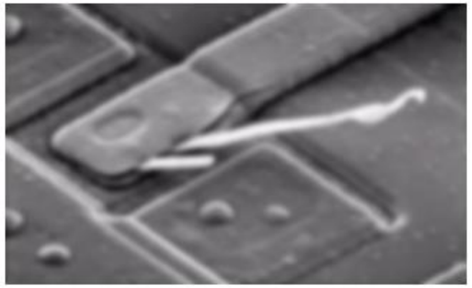
Lowpass Filter

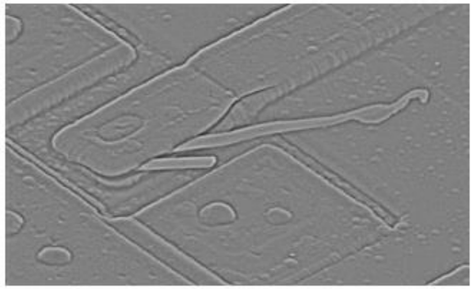




Highpass Filter

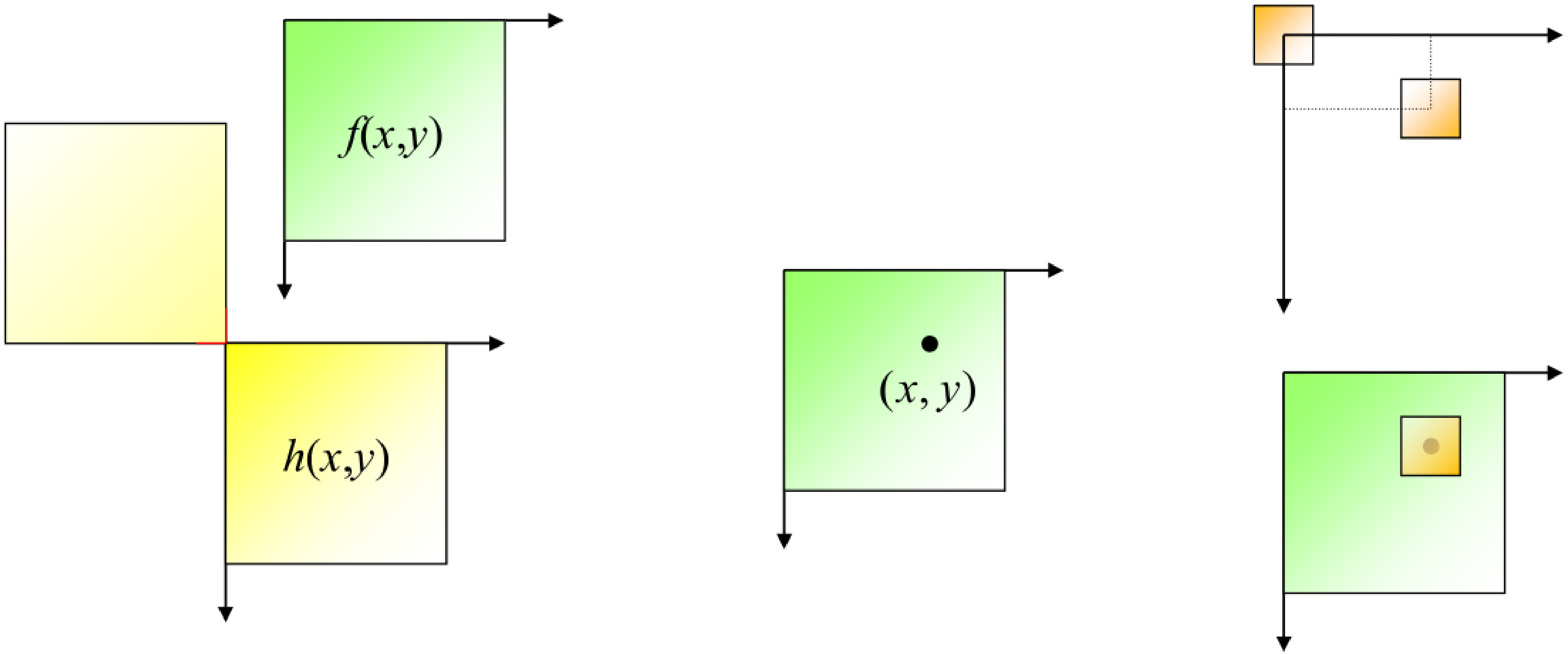






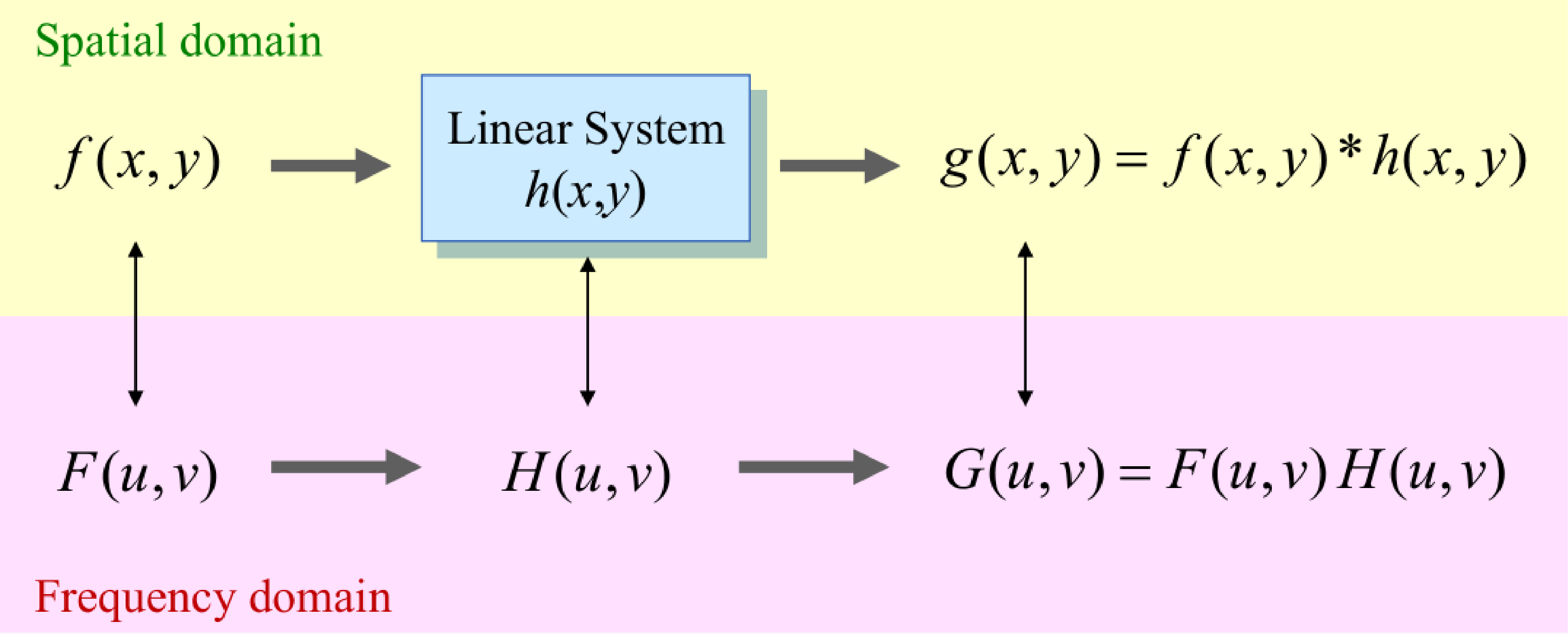
❑ Convolution

$$f(x,y)*h(x,y)=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)h(x-m,y-n)$$



❑ Convolution Theorem

$$f(x,y) * h(x,y) \Leftrightarrow F(u,v)H(u,v)$$
$$f(x,y)h(x,y) \Leftrightarrow F(u,v) * H(u,v)$$



❑ Impulse Response

$$\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\delta(x-x_0, y-y_0)s(x,y) = s(x_0,y_0)$$

$$\Delta_{x_0,y_0}(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\delta(x-x_0, y-y_0)e^{-j2\pi(ux/M+vy/N)}$$

$$= \frac{1}{MN}e^{-j2\pi(ux_0/M+vy_0/N)}$$

$$\Delta_{0,0}(u,v) = \frac{1}{MN}$$

$$\delta(x-x_0, y-y_0) * h(x,y)$$

$$= \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\delta(m-x_0, n-y_0)h(x-m, y-n)$$

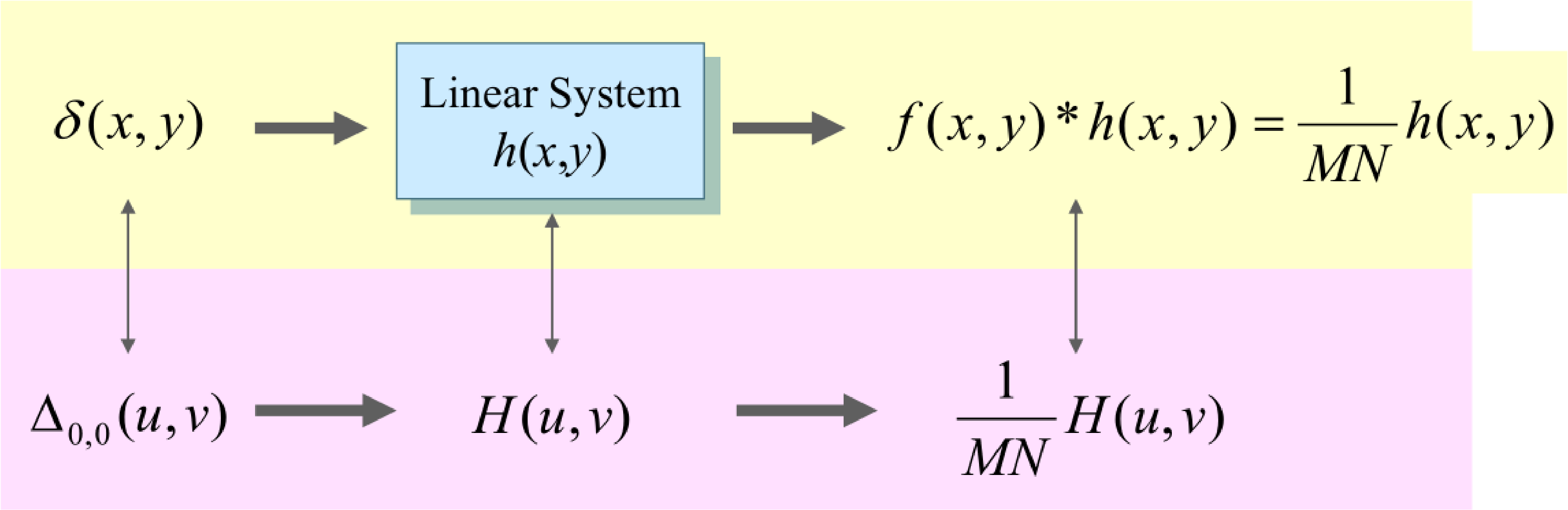
$$= \frac{1}{MN}h(x-x_0, y-y_0)$$

$$\delta(x,y) * h(x,y) = \frac{1}{MN}h(x,y)$$

Filtering in Frequency Domain

❑ Impulse Response & Convolution Theorem

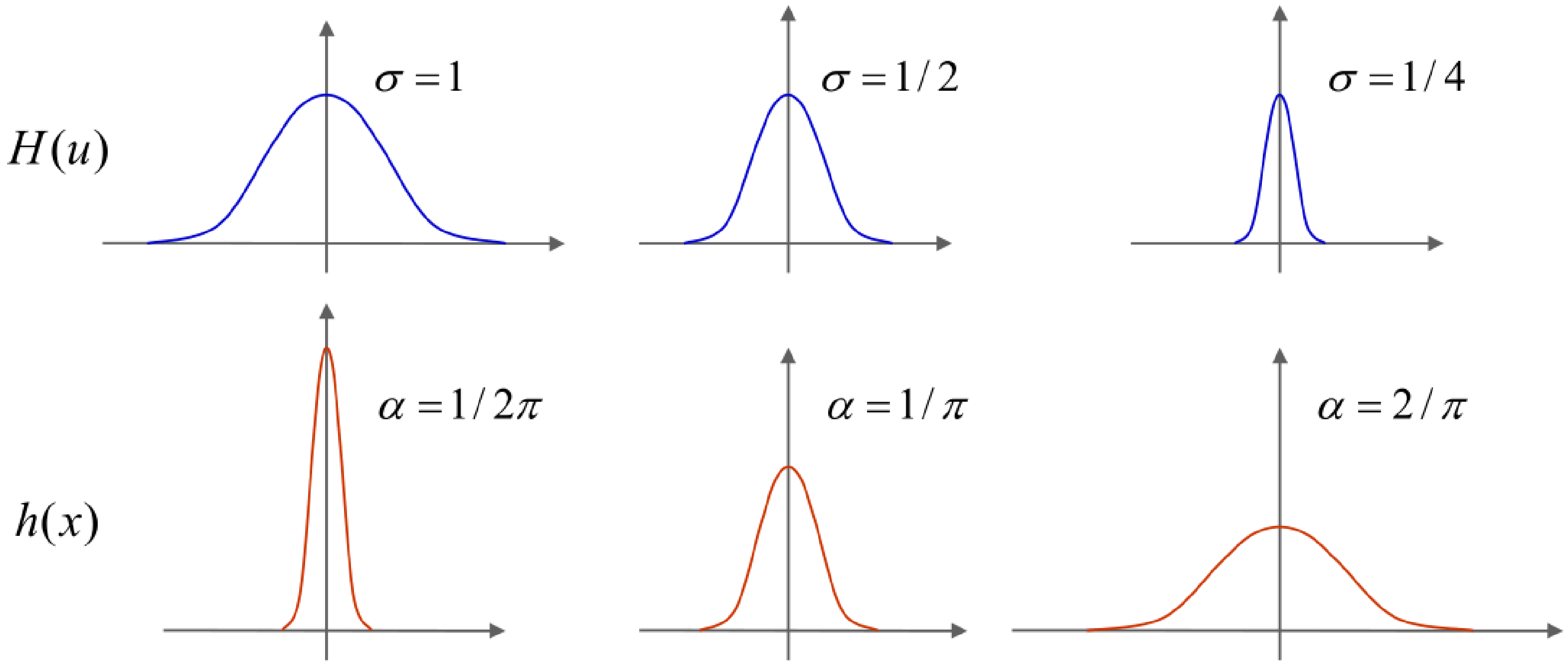
$$\delta(x,y) * h(x,y) \Leftrightarrow \Delta_{0,0}(u,v)H(u,v) = \frac{1}{MN}H(u,v)$$

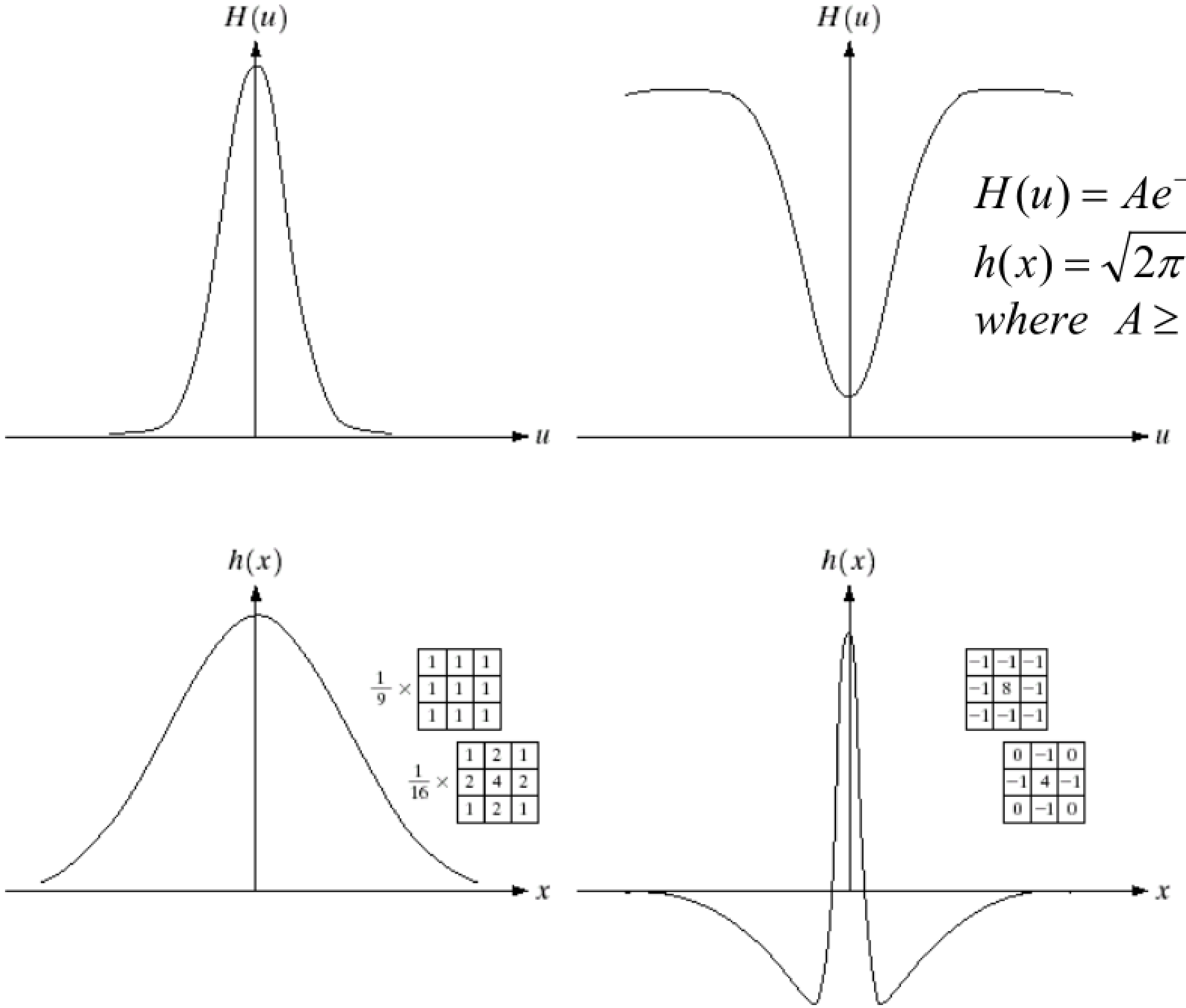


❑ Gaussian filters in spatial and frequency domain

$$H(u) = Ae^{-u^2/2\sigma^2} \Leftrightarrow h(x) = \sqrt{2\pi}\sigma\, Ae^{-x^2/2\alpha^2} \quad where \;\; \alpha = \frac{1}{2\pi\sigma}$$

$$H(u) = G(u;\sigma) \quad \Leftrightarrow h(x) = \sqrt{2\pi}\sigma\, G(x;\frac{1}{2\pi\sigma})$$





$$H(u) = Ae^{-u^2/2\sigma_1^2} - Be^{-u^2/2\sigma_2^2}$$

$$h(x) = \sqrt{2\pi}\sigma_1 Ae^{-2\pi^2\sigma_1^2x^2} - \sqrt{2\pi}\sigma_2 Be^{-2\pi^2\sigma_2^2x^2}$$

$$where \quad A \geq B \quad and \quad \sigma_1 > \sigma_2$$

Smoothing Frequency-Domain Filters

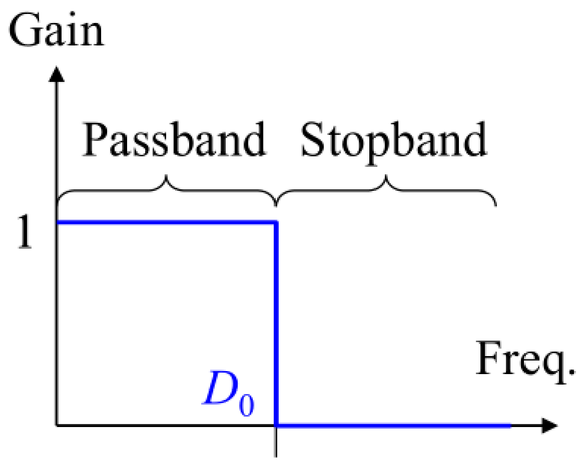
❑ Ideal Lowpass Filter

Filtering in Frequency Domain

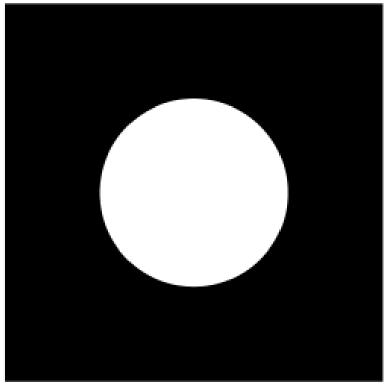
$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

Cutoff Frequency

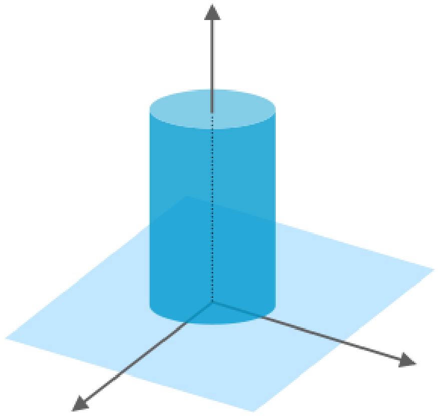
$$\text{where } D(u,v) = \left[(u - M/2)^2 + (v - N/2)^2\right]^{1/2}$$



1D Ideal Lowpass Filter



2D ILPF for Fourier spectrum



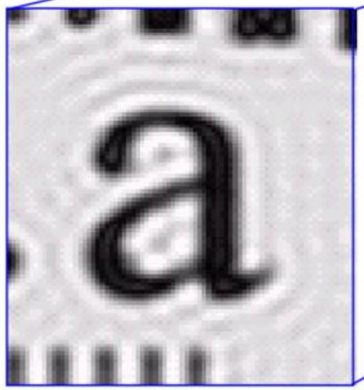
- (Example)

$$D_0 = 5, 15, 30, 80, 230$$

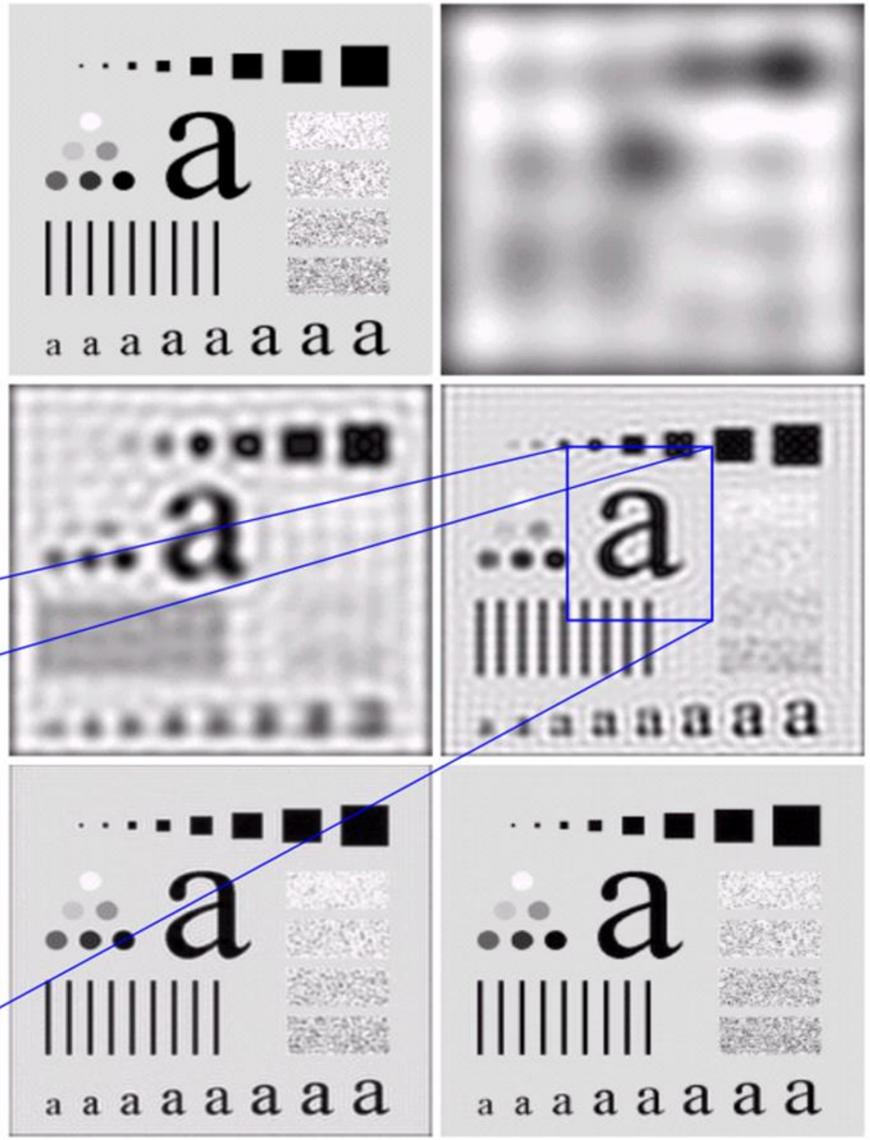
$$\alpha = 92.0, 94.6, 96.4, 98.0, 99.5\%$$

$$\alpha = \frac{100 \sum_{u,v \in C_{D_0}} P(u,v)}{\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} P(u,v)}$$

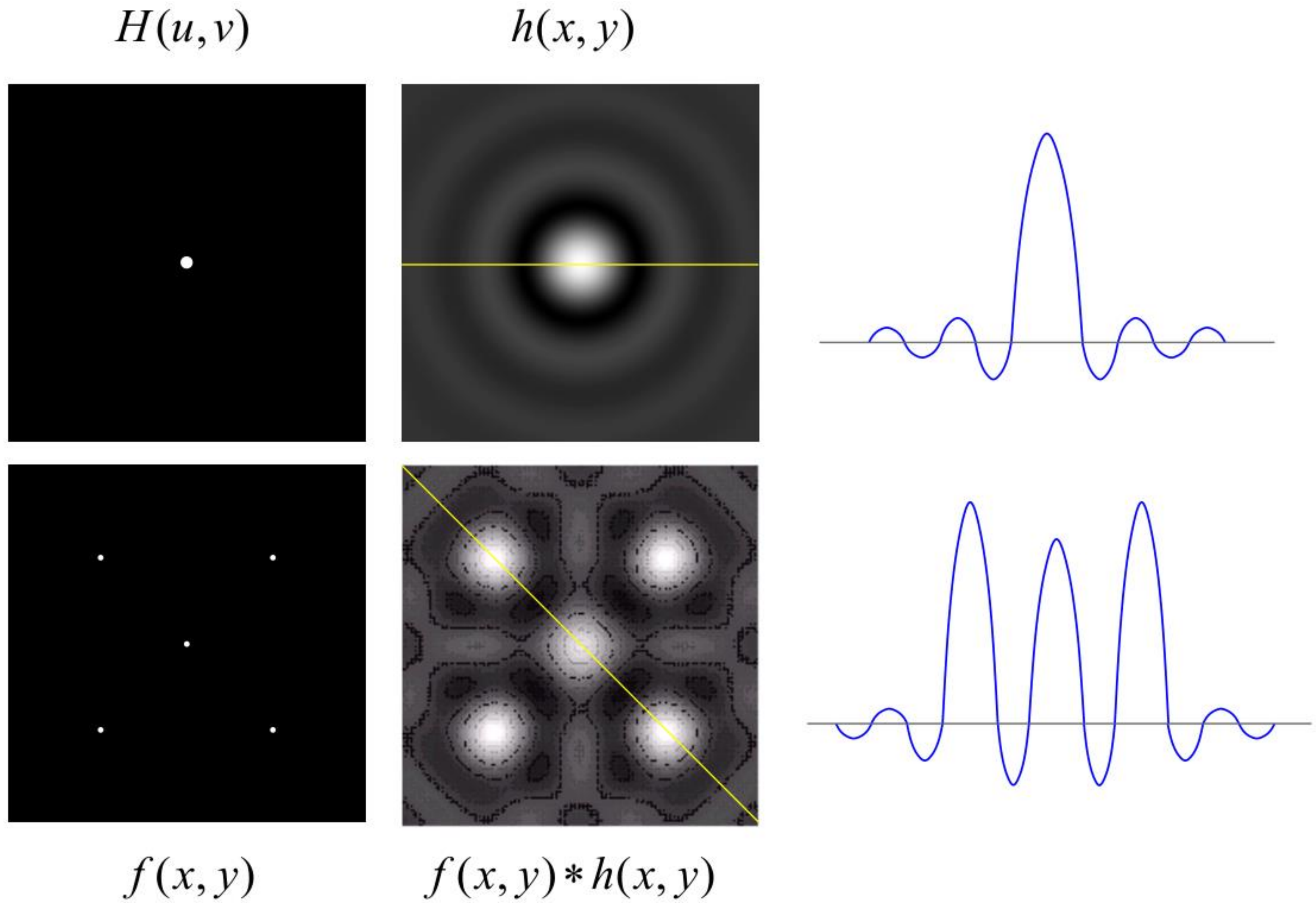
where $P(u,v)$ is the power spectrum



Ringing



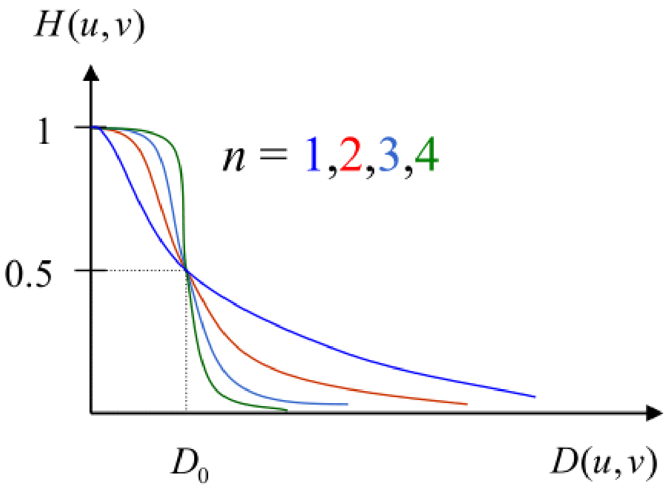
Smoothing Frequency-Domain Filters



❑ Butterworth Lowpass Filter

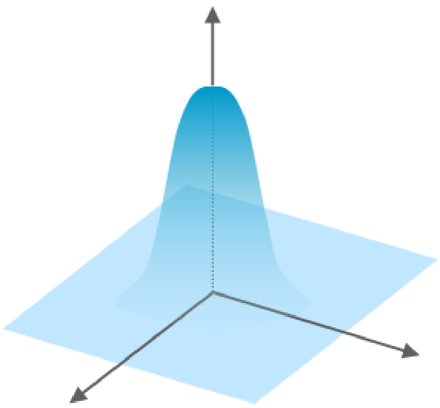
$$H(u,v) = \frac{1}{1+\left[D(u,v)/D_0\right]^{2n}}$$

$$where\ D(u,v) = \left[(u-M/2)^2 + (v-N/2)^2\right]^{1/2}$$



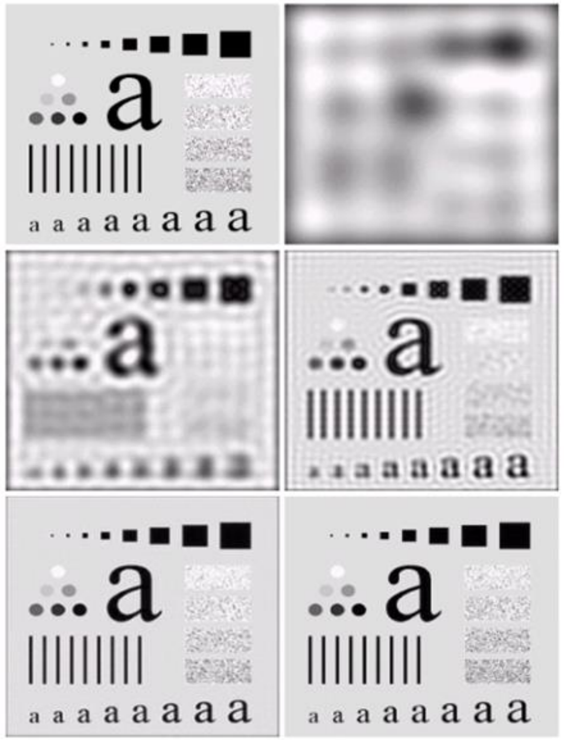
1D Butterworth Lowpass Filter

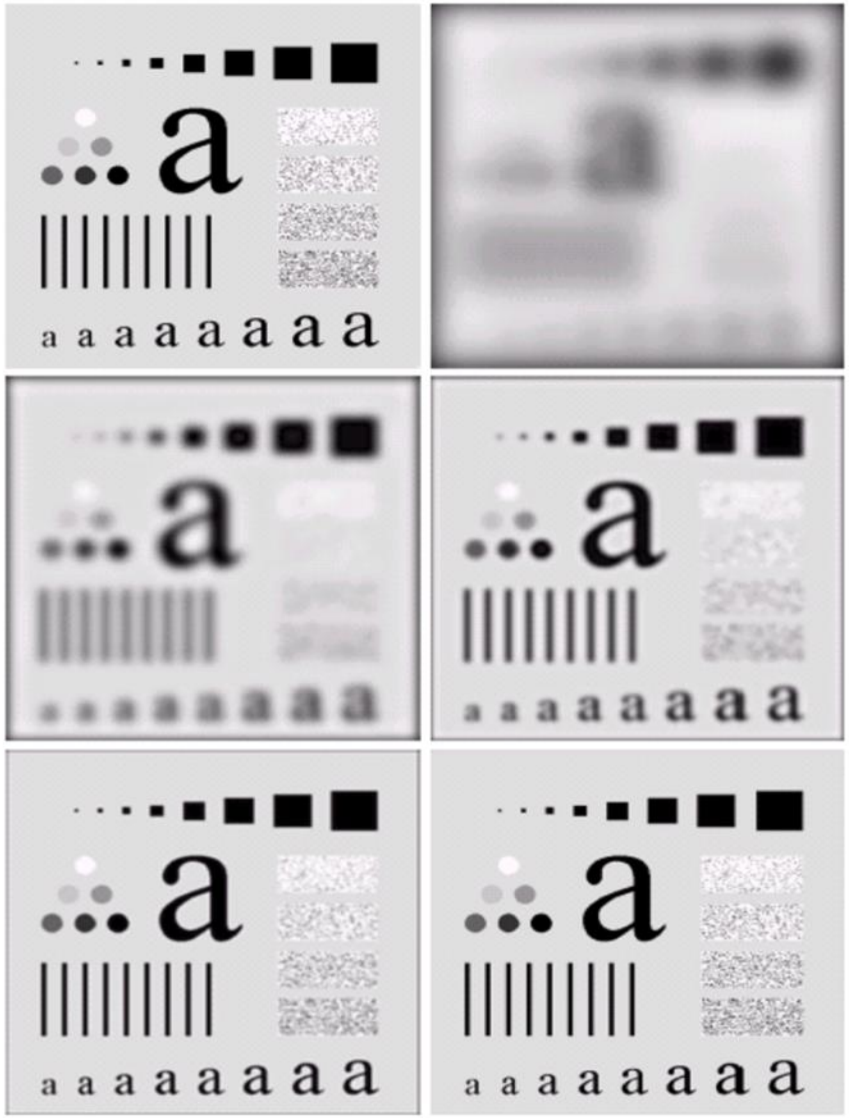
2D BLPF for Fourier spectrum



- (Example)

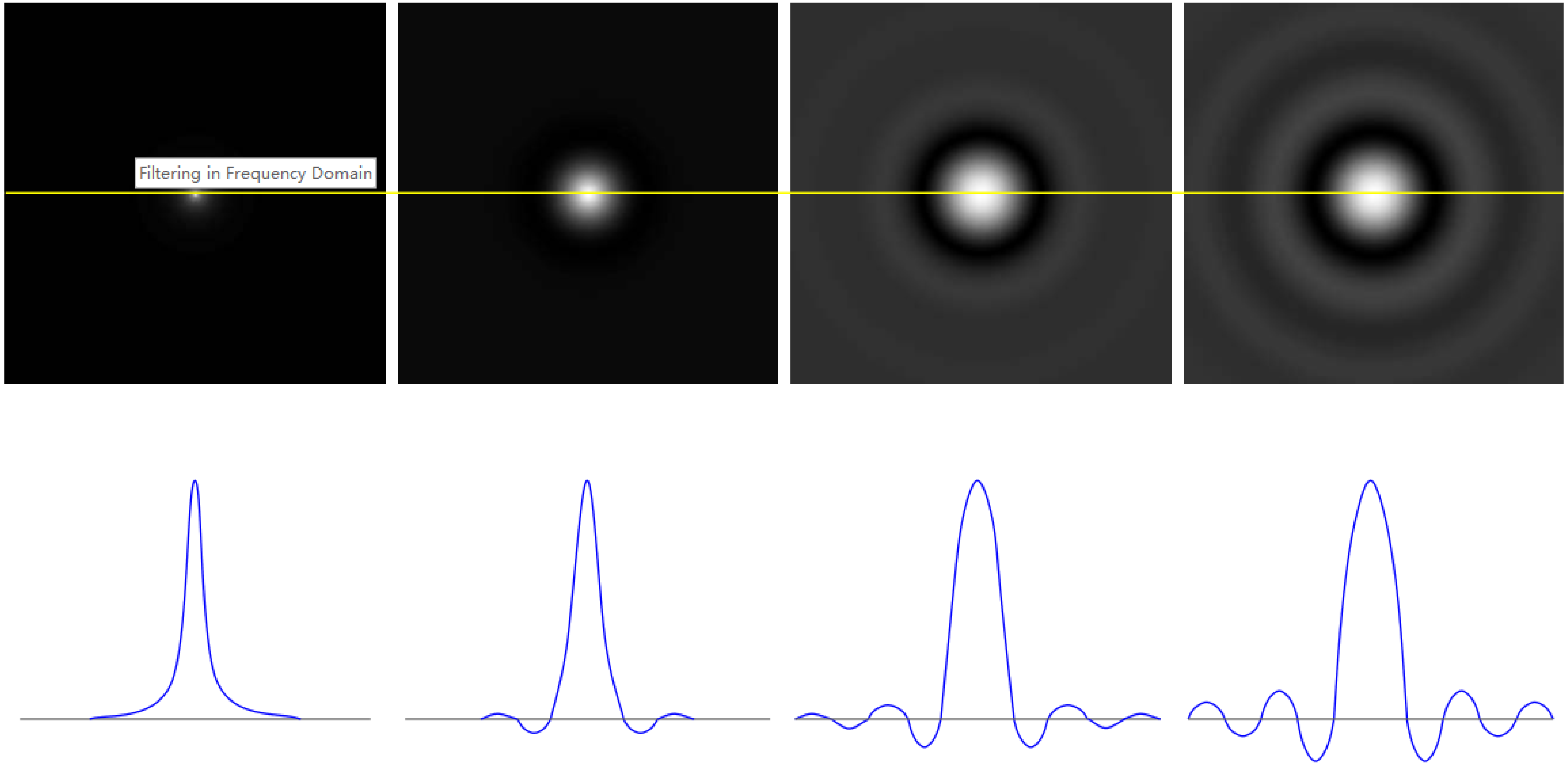
$$D_0 = 5, 15, 30, 80, 230$$





Smoothing Frequency-Domain Filters

$h(x, y)$

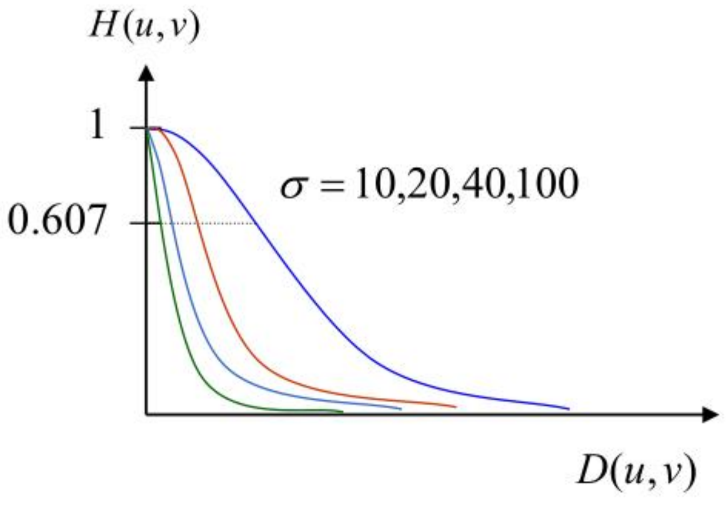


$n = 1$ $n = 2$ $n = 5$ $n = 20$

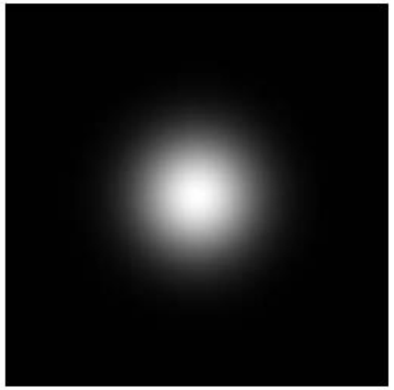
Smoothing Frequency-Domain Filters

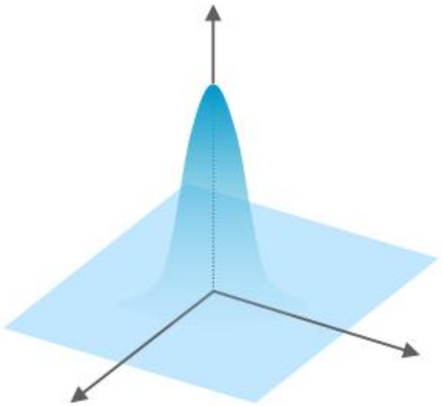
❑ Gaussian Lowpass Filter

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}$$



1D Gaussian Lowpass Filter

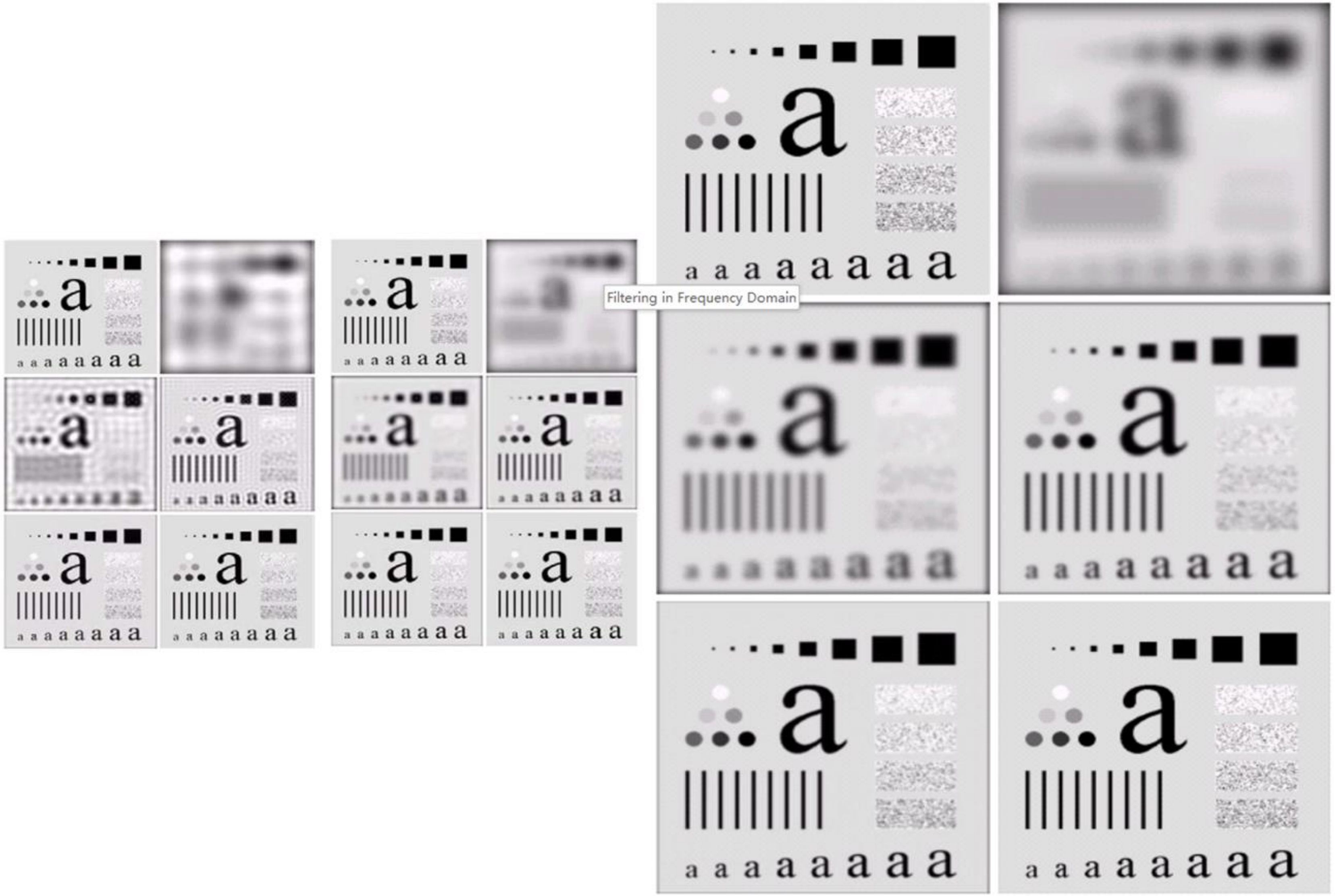




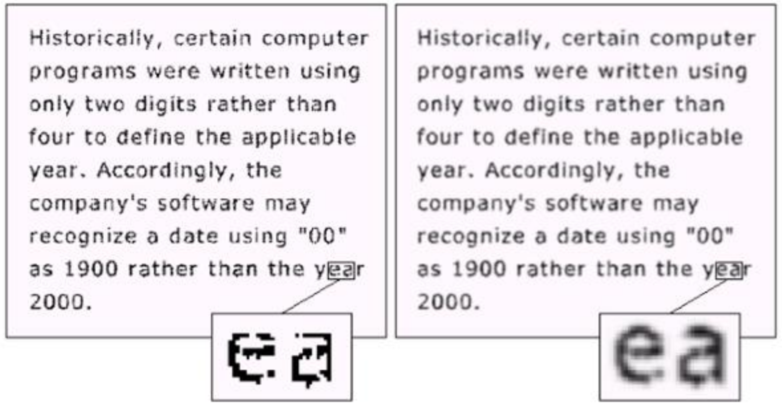
2D GLPF for Fourier spectrum

$$H(u,v)\big|_{D=\sigma} = e^{-1/2} \approx 0.607$$

- (Example)

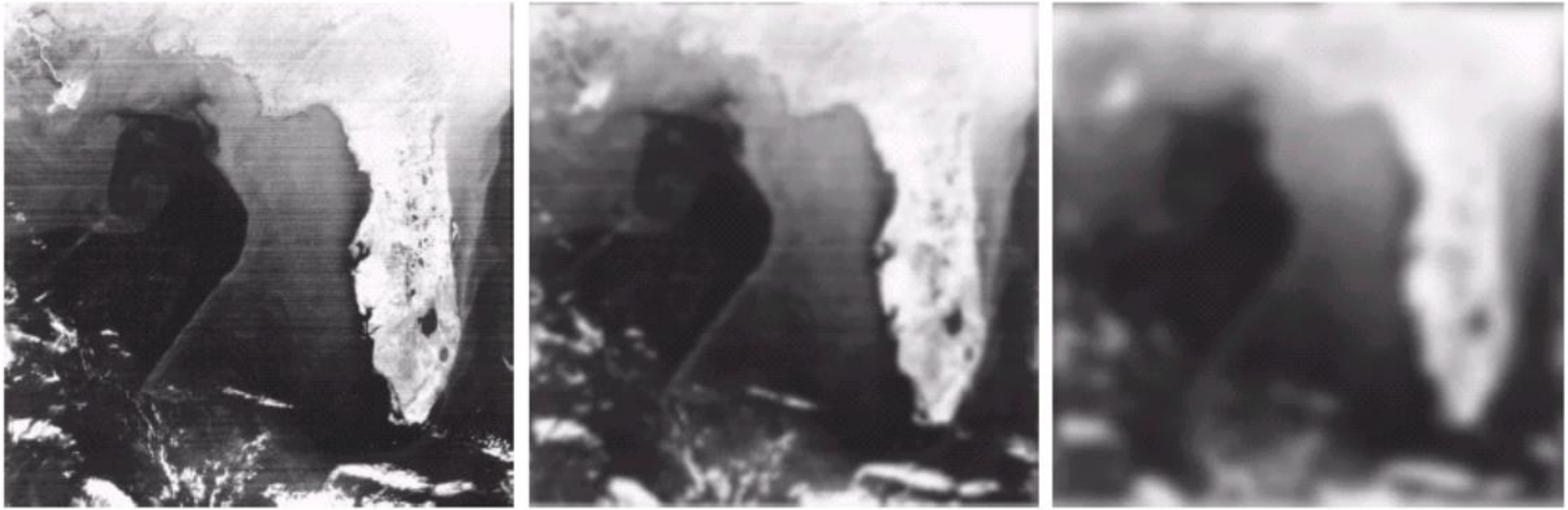


❑ Additional Examples



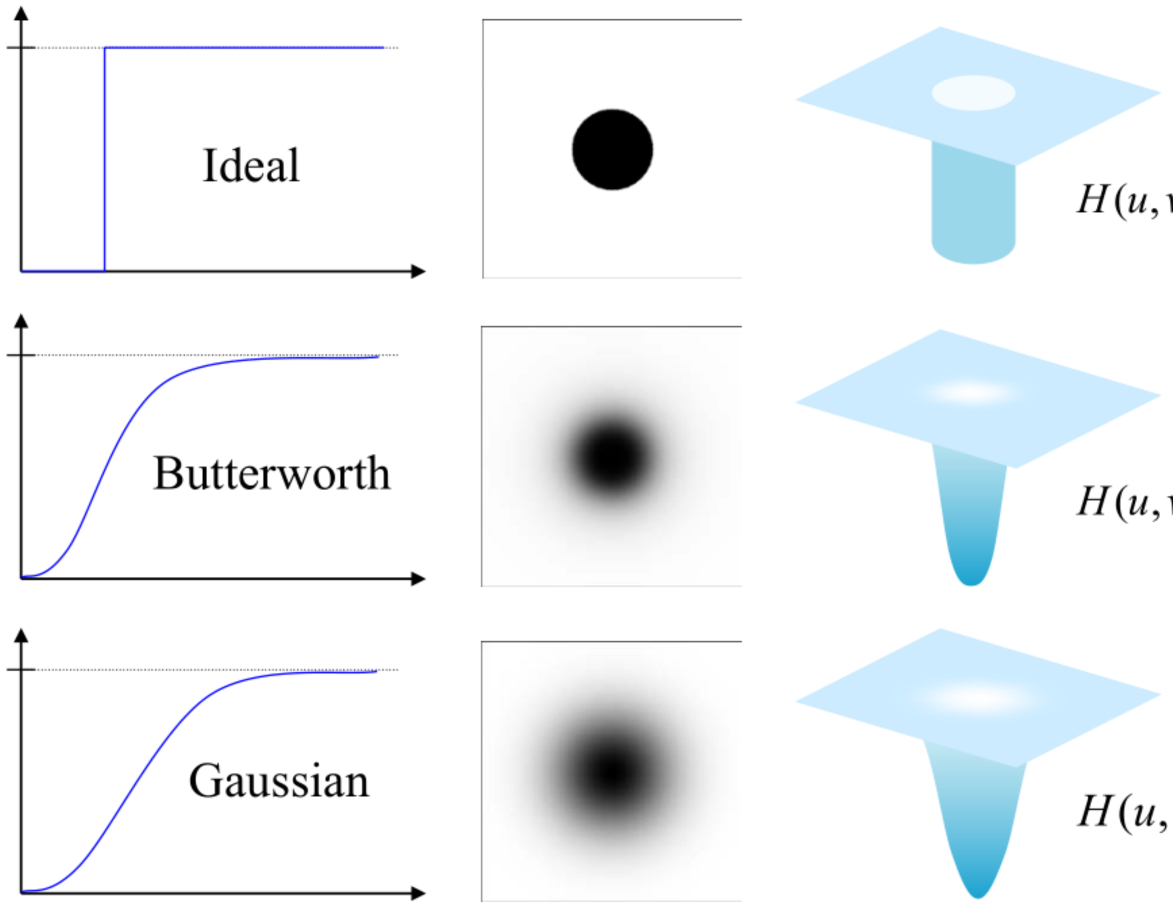
Smoothing Frequency-Domain Filters

❑ Additional Examples



Sharpening Frequency Domain Filters

$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

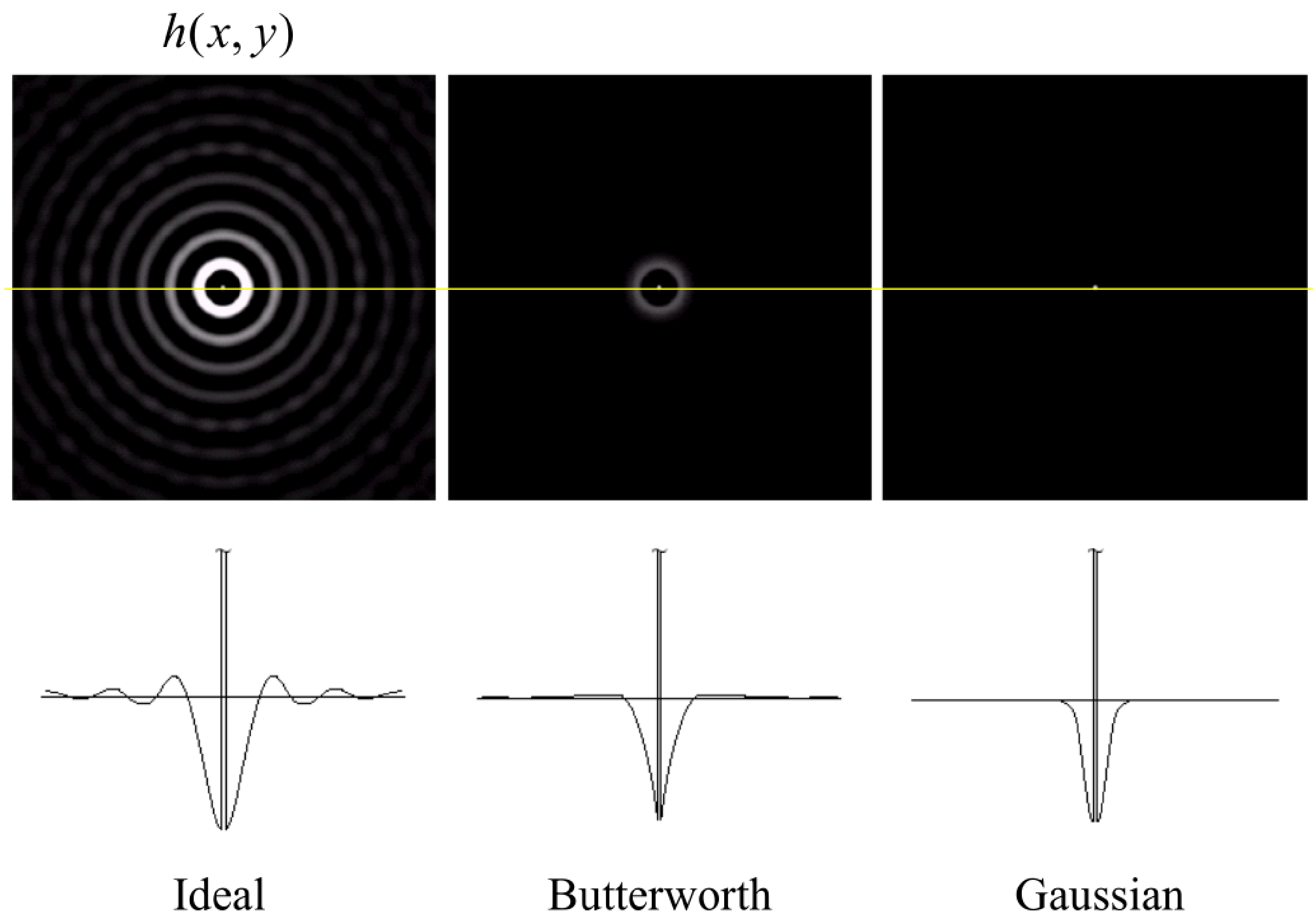


$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

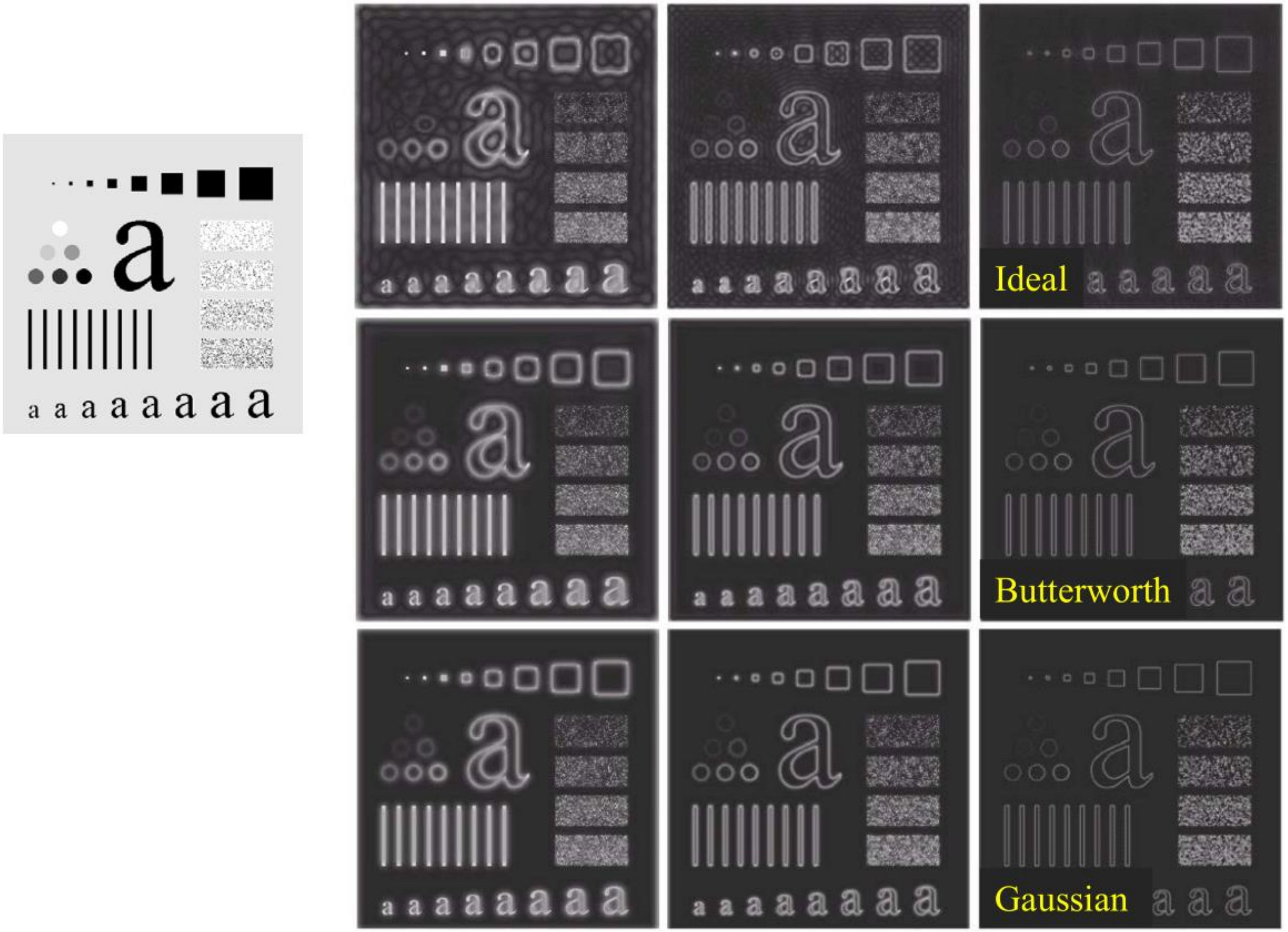
$$H(u,v) = \frac{1}{1 + [D_0 / D(u,v)]^{2n}}$$ ★

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$

Sharpening Frequency Domain Filters



Sharpening Frequency Domain Filters

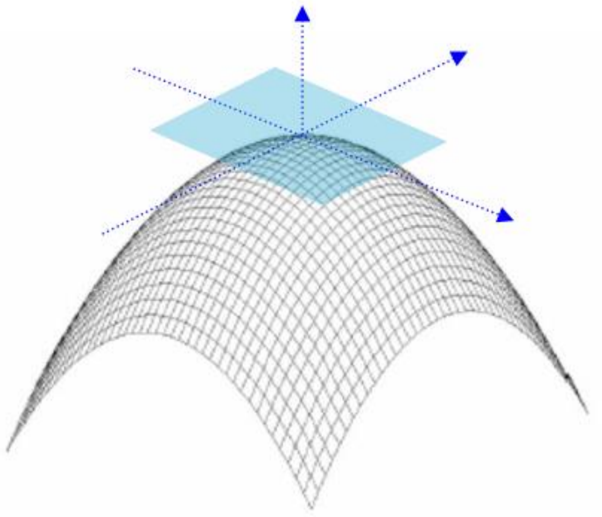


Sharpening Frequency Domain Filters

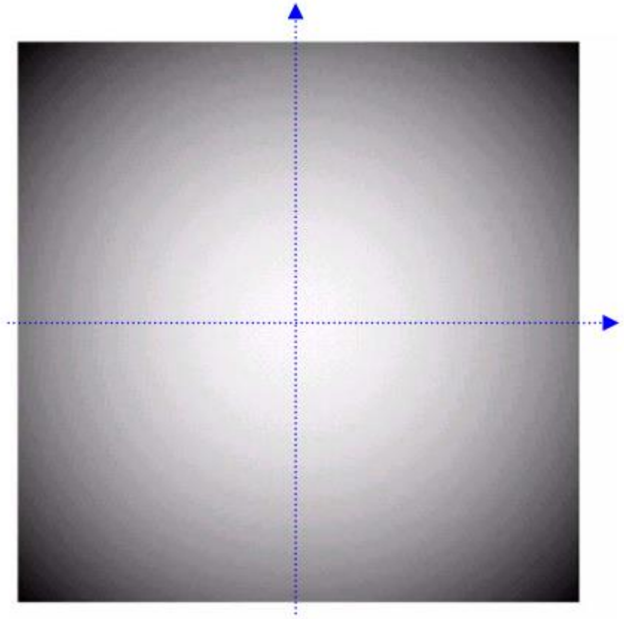
- Laplacian in frequency domain

$$\Im\left[\nabla^2 f(x,y)\right] = \Im\left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right] \quad \Leftarrow \quad \Im\left[\frac{d^n f(x)}{dx^n}\right] = (ju)^n F(u)$$

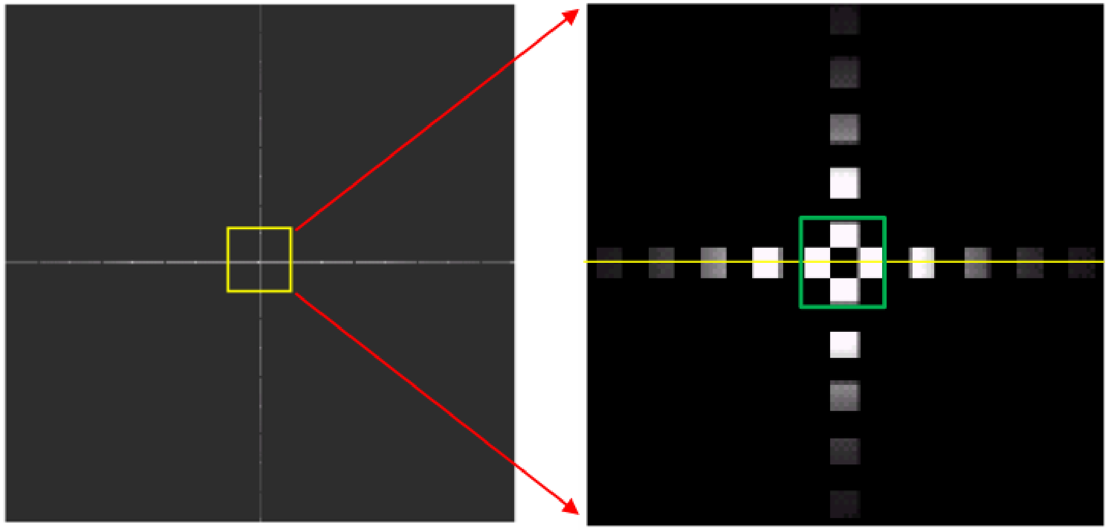
$$= (ju)^2 F(u,v) + (jv)^2 F(u,v) = -(u^2 + v^2) F(u,v)$$



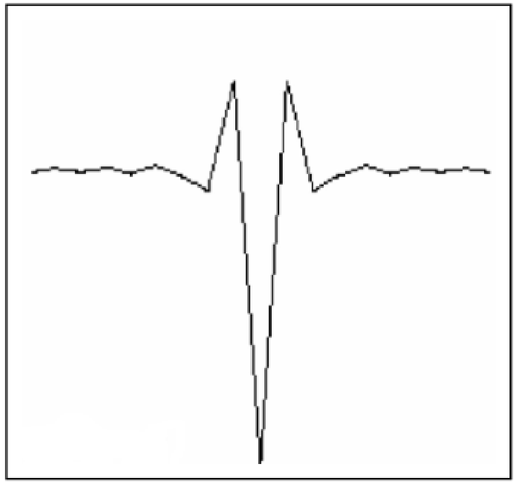
$$H(u,v) = -(u^2 + v^2)$$



Sharpening Frequency Domain Filters



$$h(x, y) = \Im^{-1}(-(u^2 + v^2))$$



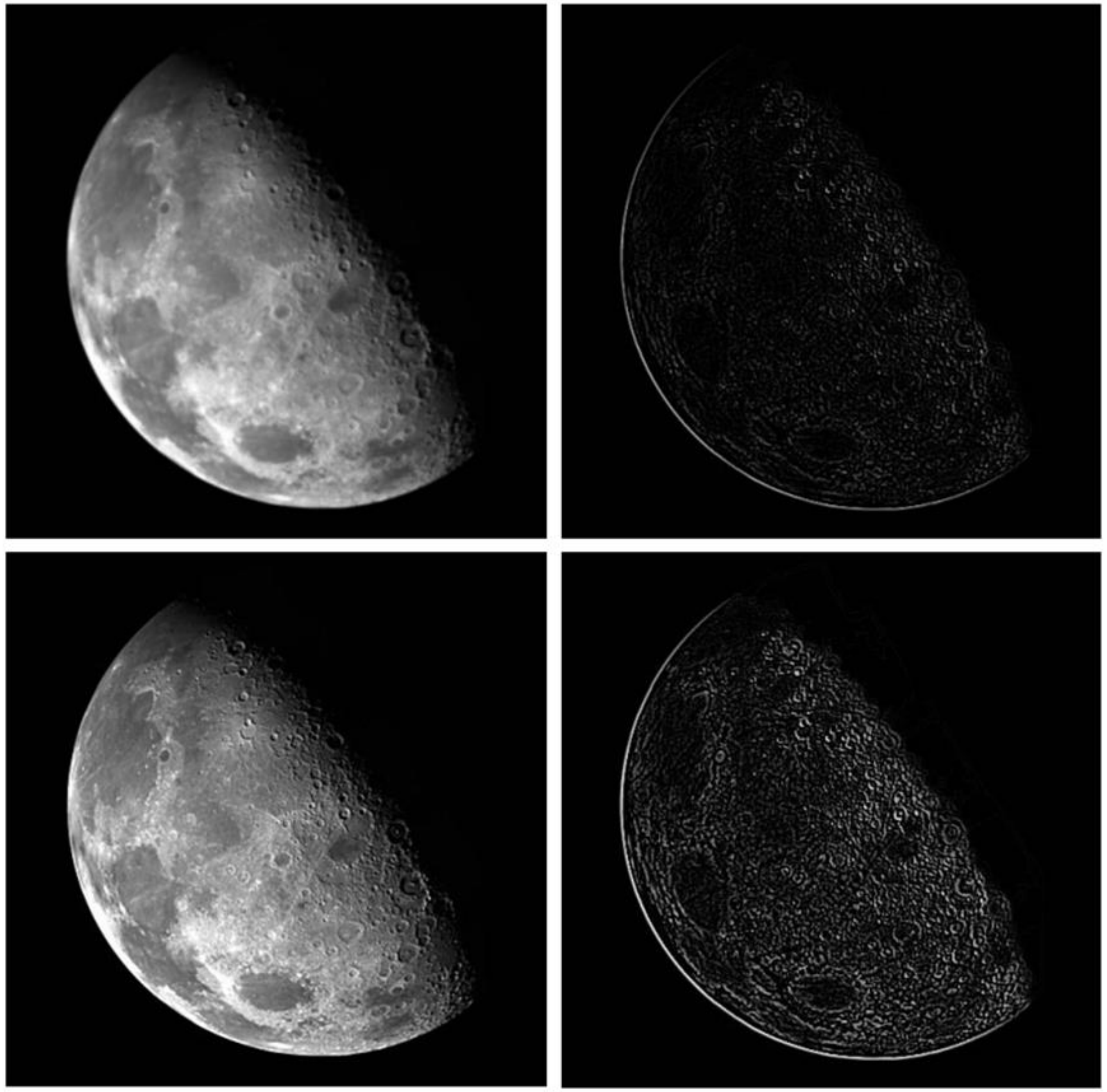
0	1	0
1	-4	1
0	1	0

- (Example)

$$g(x,y) = f(x,y) - \nabla^2 f(x,y)$$

$$G(u,v) = F(u,v) + (u^2+v^2)F(u,v) = \left[1+(u^2+v^2)\right]F(u,v)$$

$$H(u,v) = 1 + (u^2+v^2)$$



❑ Unsharp Masking and Highpass Filtering

- Unsharp masking generates a sharp image by subtracting from an image a blurred version of itself
- Highpass filtering can be considered as an unsharp masking

$$f_{usm}(x,y) = f(x,y) - k f_{lp}(x,y)$$

$$\begin{aligned} F_{usm}(u,v) &= F(u,v) - kF_{lp}(u,v) = F(u,v) - kH_{lp}(u,v)F(u,v) \\ &= (1-k)F(u,v) + k\left[1 - H_{lp}(u,v)\right]F(u,v) \\ &= (1-k)F(u,v) + kH_{hp}(u,v)F(u,v) \end{aligned}$$

$$H_{usm}(u,v) = (1-k) + kH_{hp}(u,v)$$

$$F_{hp}(u,v) = H_{hp}(u,v)\,F(u,v) \qquad ;\; k=1$$

❑ High-Boost Filtering

- Generalization of unsharp masking

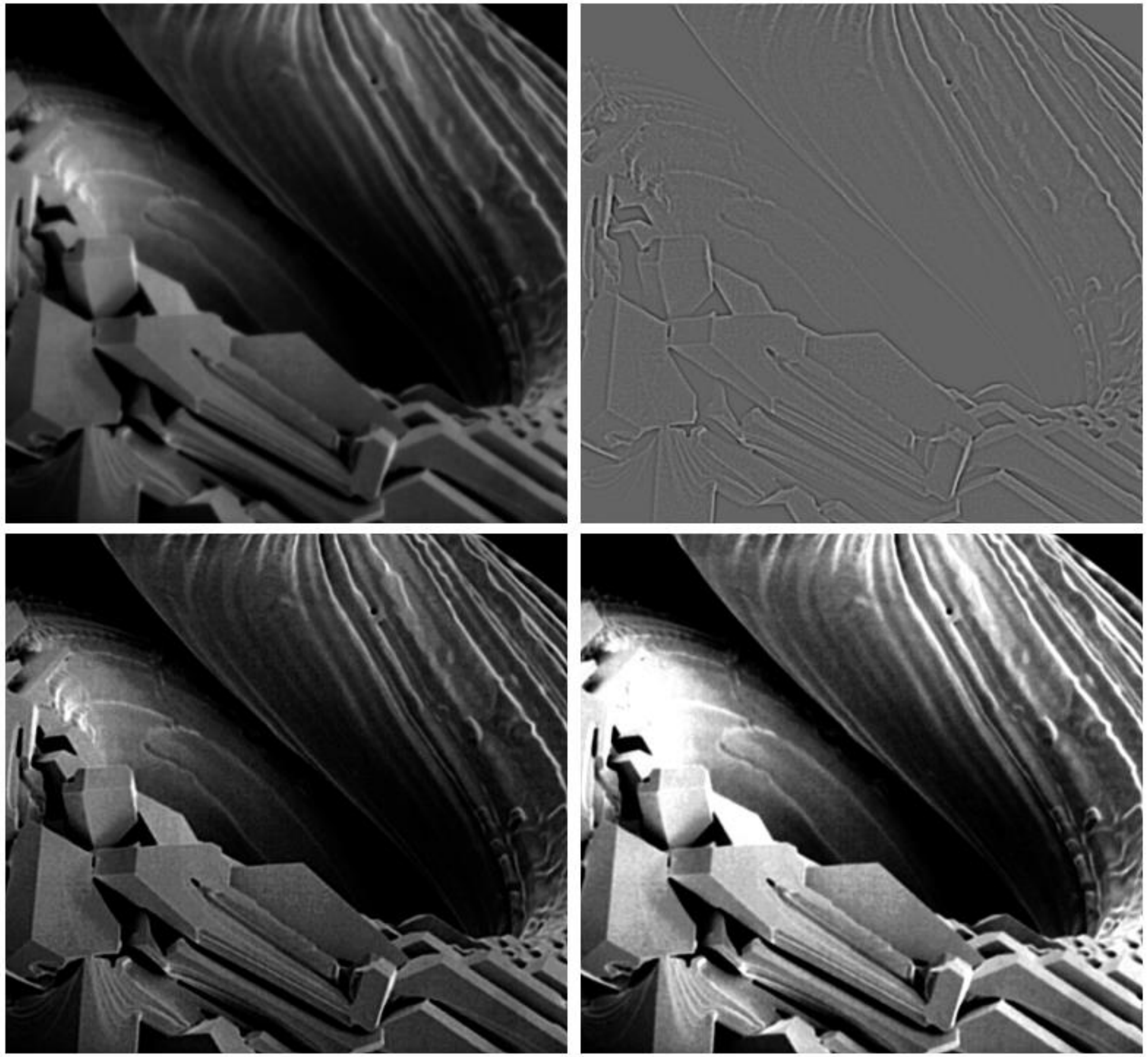
$$f_{hb}(x,y) = Af(x,y) - f_{lp}(x,y)$$
$$= (A-1)f(x,y) + f_{hp}(x,y)$$

$$F_{hb}(u,v) = (A-1)F(u,v) + H_{hp}(u,v)F(u,v)$$
$$= \left[(A-1) + H_{hp}(u,v)\right]F(u,v)$$

$$H_{hb}(u,v) = (A-1) + H_{hp}(u,v)$$

$$H_{usm}(u,v) = (1-k) + kH_{hp}(u,v)$$

Sharpening Frequency Domain Filters



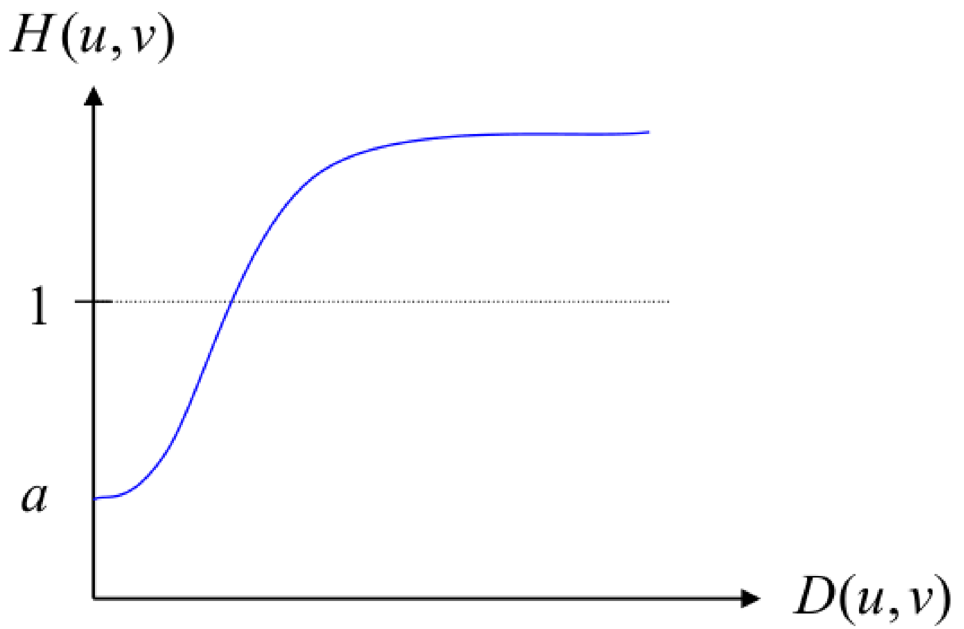
$A = 1$

$A = 2$

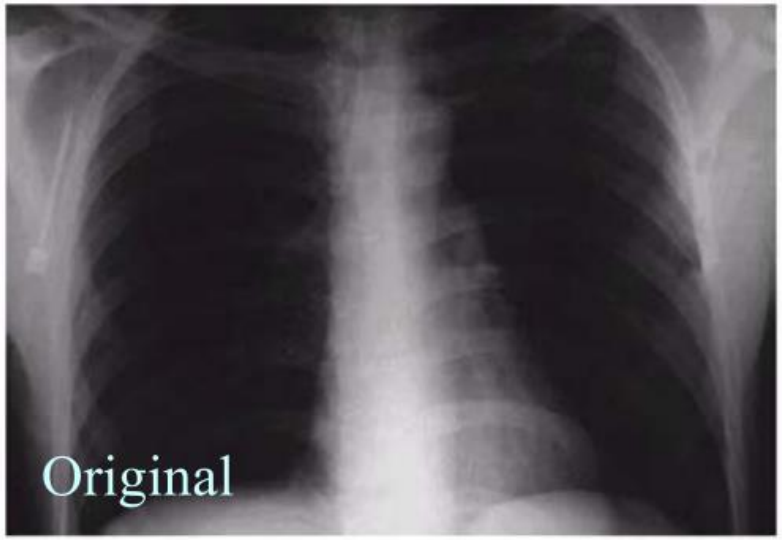
$A = 2.7$

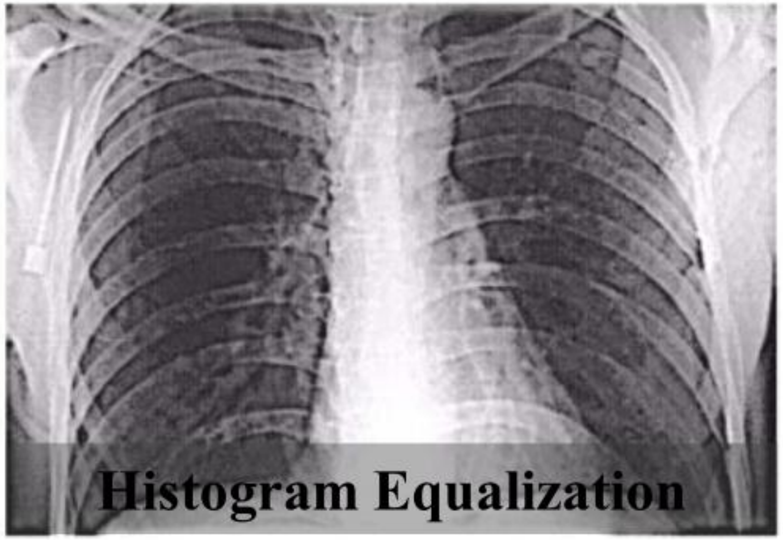
❑ High Frequency Emphasis Filter

$$H_{hfe}(u,v) = a + bH_{hp}(u,v)$$
$$where\ \ 0.25 \le a \le 0.5\ \ and\ \ 1.5 \le b \le 2.0\ \ typically$$



Sharpening Frequency Domain Filters





Summary

- Fourier Transform
- Filtering in Frequency Domain
- Smoothing Frequency Domain Filters
- Sharpening Frequency Domain Filters

- Next: Image Restoration and Reconstruction

Q & A

Thank You!